Lecture Notes in Macroeconomics

Growth, part 1

Miscellaneous Preliminaries

What is a production function?

A production function is a mathematical function that tells us how much output we get for given amounts of inputs. For the time being, we are going to have just two inputs, K and L, and so we will write the production function as

\[ Y = F(K, L) \]

At times it will be convenient to have a specific production function to think about. So for today we will think about a very useful one, the Cobb-Douglas production function:

\[ Y = K^\alpha L^\beta \]

Definition of returns to scale. If have a production function

\[ Y = F(A, B) \]

then we ask what happens if we multiply both factors of production by some factor \( z \), that is, what is \( F(zA, zB) \)?

If the answer is that output gets multiplied by \( z \) also, then we say that the production function has constant returns to scale. That is

\[
\begin{align*}
\text{CRS:} & \quad zY = F(zA, zB) \\
\text{IRS:} & \quad zY < F(zA, zB) \\
\text{DRS:} & \quad zY > F(zA, zB)
\end{align*}
\]

Suppose we want to know whether a cobb-douglas production function is constant returns to scale. Then we just do

\[
F(zK, zL) = (zK)^\alpha (zL)^\beta = z^{\alpha+\beta} K^\alpha L^\beta = z^{\alpha+\beta} Y
\]
so if \( \alpha + \beta = 1 \), then the production function is CRS.

**Marginal Products**

Recall that the marginal product of a factor of production is the amount of extra output that we get for one extra unit of input.

\[
MPK = \frac{dY}{dK}
\]

this is also written as \( F_K(K,L) \) or just \( F_K \)

Marginal products for cobb douglas \( Y = K^\alpha L^\beta \)

\[
MPL = \frac{dY}{dL} = \beta K^\alpha L^{\beta-1}
\]

\[
MPK = \frac{dY}{dK} = \alpha K^{\alpha-1} L^\beta
\]

One of the properties demonstrated by most production functions is diminishing marginal product.

\[
\frac{d(MPL)}{dL} \leq \frac{d^2Y}{dL^2} = F_{LL} < 0
\]

This says that if you hold the amount of one factor constant, and increase the amount of the other factor, then the marginal product of the second factor will fall.

[picture -- with output on Y axis, L on x axis, K held constant]

here the marginal product of labor is just the slope of this line.

Looking at the Cobb Douglas marginal products, we can confirm that it is true:

\[
F_{LL} = \beta(\beta-1)K^{\alpha}L^{\beta-2} < 0
\]

**Why factors get paid their marginal products**
In our circular flow picture, we have firms paying for both labor and for capital, the two factors of production. One question to ask is how is it determined how much is paid to each factor?

To see how the prices of each factor (the wage for labor, the rental rate for capital) are determined, we have to look at firms, which hire these factors of production.

We start by examining the firm's production function. We assume that firms are competitive and small and that the economy-wide production function, which is the same as the firm's production function, is CRS.

\[ Y = F(K,L) \]

We assume that firms take the wage and the rental rate as given, and we want to find their demand functions for each factor.

The firm is trying to maximize profits:

\[ \text{profit} = Y - wL - rK \]

note that we assume that the price of output is 1, and \( w \) and \( r \) are the wage and the rental rates.

Well, we know the two first order conditions (from taking the derivatives with respect to \( L \) and to \( K \), then setting equal to zero):

\[ \frac{dY}{dL} = w \]
\[ \frac{dY}{dK} = r \]

These first order conditions tell us the relation between the marginal products of capital and labor and the wage rate.

So the assumption that firms are maximizing profits guarantees that factors are paid their marginal products.

**Factor exhaustion** (Euler's theorem)

Now we ask the question: if labor and capital are paid their marginal products, is there any output left over? Luckily, the answer is that if production is CRS, then payments to the two factors of production **exactly** use up all of output. To prove this, just take the definition of CRS

\[ zF(K,L) = F(zK,zL) \]
now differentiate with respect to $z$

$$dz \ F(K,L) = K \cdot dz \cdot F_K(zK,zL) + L \cdot dz \cdot F_L(zK,zL)$$

cancel the $dz$ terms from each term, and then evaluate at $z=1$ (that is, this is true for any $z$, so it is true when $z=1$)

$$F(K,L) = K \cdot F_K(K,L) + L \cdot F_L(K,L)$$

QED

**Factor shares in the Cobb-Douglas production function**

For the familiar Cobb-Douglas production function we can do an even neater thing.

Take the marginal product of capital

$$MPK = \frac{dY}{dK} = \alpha K^{\alpha-1} L^{\beta}$$

So total payments to capital are $K \cdot MPK = \alpha K^{\alpha} L^{\beta} = \alpha Y$

Now divide this by total output:

$$\frac{K \cdot MPK}{Y} = \alpha$$

Similarly, if we did this exercise for labor, we find that

$$L \cdot MPL / Y = \beta$$

So for a constant returns cobb-douglas production fn (that is $\alpha + \beta = 1$), not only is it the case that payments to factors exhaust all of output, but also that the share that each factor receives is constant. So if you have more capital, the rent on capital falls just enough so that the total fraction of output that goes to capital is constant.

In fact, this appears to be roughly true, and it is one of the reasons that we think that the Cobb-Douglas production function is a good one to use.

**Solow growth model**

Start with the simplest model of growth, which we will use as a base with which to build bigger,
richer models.

Start with a production function that has two arguments: labor and capital. We will take the labor force as exogenous -- we will not worry about things like unemployment or the labor force participation rate until later.

\[ Y = F(K,L) \]

We assume that the production function is constant returns to scale. Thus we can divide everything by \( L \):

\[ \frac{Y}{L} = F\left(\frac{K}{L},\frac{L}{L}\right) \quad \text{now define } y = \frac{Y}{L} \quad \text{and } k = \frac{K}{L} \]

\[ y = F(k,1) = f(k) \]

That is, output per worker depends only on the amount of capital per worker. Note that the new production function, \( f \), is really the same as \( F \), but with the second argument set to be one.

To take a Cobb-Douglas example, suppose that the production function were

\[ Y = K^\alpha L^{1-\alpha} \]

in per-worker terms it would be

\[ y = k^\alpha \]

For our first cut, we will make things simple by assuming that the population is constant and that the production function does not change over time.

The only thing that we have to think about is how \( k \) changes over time.

Well, let's go back to stocks and flows. \( K \) is the capital stock. What determines how it changes?

Two factors: investment and depreciation.

Investment is the flow of new capital. Call \( i \) the amount of investment per period (in per-worker terms)

But there is another factor: every period some of the capital stock depreciates. Call \( d \) the amount of depreciation per capita each period.

Note that both \( i \) and \( d \) are flows, while \( k \) is a stock.
So \( \frac{dk}{dt} = i - d \)

Note that from now on, we will write time derivatives with a dot on top of them: \( \frac{dk}{dt} = \dot{k} \)

For investment, we assume that a constant fraction of output, \( s \), is saved, and that this a closed economy so that saving equals investment: \( i = sy = s^*f(k) \)

For depreciation, we assume that a constant fraction of the capital stock, \( \delta \), depreciates each period: \( d = \delta k \)

So the equation for the change in capital over time is

\[
\dot{k} = sf(k) - \delta k
\]

This is a differential equation for the evolution of the per-person capital stock.

Now we ask the question: what is the level of the capital stock such that the capital stock is constant over time? We call this level of the capital the **steady state**.

Well, it should be clear that at the steady state, \( k^* = 0 \). That is, the flow of new capital from investment is exactly the same as the destruction of capital from depreciation.

so we define \( k^* \) as the level of capital such that \( s \times f(k^*) = \delta \times k^* \)

We can draw a picture of the two sides of this equation to get some idea of what is going on:

[picture]

Note that when \( k \) is below the steady state level \( \dot{k} \) is positive, so that the amount of capital in the economy is increasing. Similarly, when \( k \) is above the steady state, etc. So the steady state is **stable**.

One thing that we can do is use this picture to see how changes in the saving rate or the rate of depreciation affect the steady state level of capital. For example, if we raise the rate of saving we shift the \( s \times f(k) \) curve. Similarly, if we change the rate of depreciation, we shift the \( \delta k \) line.

To do a more concrete example, let's use a particular production function: the Cobb-Douglas production function that we looked at before:

\[
f(k) = k^\alpha
\]
so the condition that $s \times f(k^*) = \delta \times k^*$ becomes:

$$s \times k^\alpha = \delta \times k$$

which solves to:  

$$k^* = \frac{(s/\delta)^{1/(1-\alpha)}}{1/(1-\alpha)}$$

So this is the steady state level of capital. What about output? Well clearly there is a steady state level of output:

$$y^* = f(k^*) = \frac{(s/\delta)^{\alpha/(1-\alpha)}}{1/(1-\alpha)}$$

So this tells us how the steady state amount of output depends on the production function and the rates of saving and depreciation. Note that steady state output does not depend on your initial level of output or your initial capital stock.

The Golden Rule for Saving

So far we have been taking the saving rate as given. That is, as exogenous to the model. In the first part of the course we talked about the determinants of saving, and we will get back to this question later in this section. But now we will start asking questions not only about what the model produces, but about what is optimal. For now we will ask a simple question: What is the level of capital that maximizes consumption in the steady state? We will call this level of the capital stock the golden rule level of the capital stock. (See the Phelps article).

Here we can draw a picture or apply a little calculus. Picture: draw a picture with $f(k)$ and $\delta k$. By choosing different saving rates you could make any level of capital less than the level where $f(k) = \delta k$ be the steady state capital stock. At any level of capital, the amount of consumption is just the distance between total output and the amount being used to replace depreciated capital. It should be clear that the distance between these two curves is greatest when the slopes are equalized. The slope of the depreciation line is just $\delta$, and the slope of the production function is $f'(k)$. So the golden rule level of capital is the level where:

$$f'(k) = \delta$$

Using calculus: we want to maximize $c = (1-s) f(k)$
s.t. $s \times f(k) = \delta \times k$

That is, we want the optimal levels of $s$ and $k$, but there is the constraint that we have to be in a steady state. One way to solve this is to just plug the constraint into the problem

$$\max c = f(k) - \delta \times k$$

take derivative: $f'(k) = \delta$

So we define $k^{**}$ as the golden rule level of the capital stock, the level of the capital stock which maximizes consumption.

Now note a very interesting property:

We know that the rent on capital in this economy is $f'(k)$ -- that is the marginal product of capital. Total payments to capital are $k \times f'(k)$ -- that is the rent per unit of capital times the marginal product of capital. Now, suppose that all of the return to capital were saved, and all of the return to labor were consumed. Then

$$s = k \times f'(k)/f(k) \implies s \times f(k) = k \times f'(k)$$

and in the steady state: $s \times f(k) = \delta \times k$

so $\delta \times k = k \times f'(k)$

$$\implies \delta = f'(k)$$

In other words, if all of the rent on capital were invested and all of labor's wages were consumed, we would end up at the golden rule level of capital!!!! Pretty amazing.

Once again, we can do the example using the Cobb-Douglas production function, even more easily.

The condition for being at the golden rule is

$$\delta = f'(k) = \alpha k^{\alpha-1} \implies k^{**} = (\alpha/\delta)^{(1/(1-\alpha))}$$

and the steady state condition is

$$k^* = (s/\delta)^{(1/(1-\alpha))}$$
So comparing the expressions for the steady state and the golden rule, it should be clear that the saving rate that sets them equal is \( s = \alpha \). But remember that we already knew that, for a Cobb-Douglas production function, capital's share of output is equal to \( \alpha \).

One question to ponder: how do we know if we are above or below the golden rule level of capital? Would it be possible to imagine an economy that somehow does have more capital than the golden rule, but in which everyone were nonetheless acting rationally? We shall take this up later on.

Dynamics: the response of output and consumption to changes in the saving rate. Suppose the economy starts out above the golden rule and then the saving rate is reduced to the GR level. Suppose that it started out below and the same thing happened.

Example: suppose that we were in steady state with \( s > \) golden rule level. Now we switch to \( s \) equal to Golden Rule level. Trace out the path of consumption.

Introducing population growth

Now that we have established intuition in the basic model, we can make it a little more realistic. We do this first by introducing population growth.

Start in non-per capita terms (i.e. capital letters)

\[
\dot{K} = sK - \delta K
\]

Now we introduce population growth at rate \( n \). That is, let

\[
n = \frac{\dot{L}}{L}
\]

that is, \( n \) is the percentage rate of increase in population

Now we just take \( K/L \) and differentiate with respect to time:
Intuitively, what is going on here is that as the population grows, the amount of capital per person falls. So population growth is just like depreciation in the way it affects the amount of capital per person.

[Note that population growth also affect the level of population, which in turn affects the amount of fixed resources (ie land) per person. Here we assume that all resources are reproducible (ie capital and labor). If there really are fixed resources that are important, then there will not be constant returns to reproducible factors (ie capital and labor).]

The rest of the analysis now goes through exactly as before:

steady state: $s f(k^*) = (n+\delta)k^*$

golden rule: $f'(k^{**}) = (n+\delta)$

[To be added: quantitative exercise. For a reasonable production function and variation in $s$, how big is the variation in $y$ that we get?]

Question: we have called this a growth model. Is output growing? Answer: yes, sort of. Output per worker is constant in steady state, but total output is growing at the rate of population growth, $n$. (Similarly, the capital stock is growing at $n$). It explains why high population growth is associated with low income. And it has transitional dynamics, for example in response to a fall in $n$. Still, this does not match the facts in the real world, since in the real world output per worker is also growing.
So we have to add in ..... 

**Technical Change**

We now alter the model to accommodate technical change. This turns out to be a can of worms in general, but we look at a simple case. In particular, we imagine that the technical change takes the form of increased efficiency of workers. That is, we say the production function is now

\[ Y = F(K, eL), \]

where \( e \) is the efficiency of each worker. \( eL \) is the number of efficiency units in the economy, and this, rather than \( L \) (the number of workers) is what is important for production.

Further, we say that the number of efficiency units is growing over time at some rate \( g \). That is

\[ g = \frac{\dot{e}}{e} \quad \text{g is the rate of improvement of technology.} \]

(This form of technical change is called "labor augmenting" or "Harrod neutral." Another commonly used specification is to have technology enter as a coefficient in front of the production function: \( Y = A*F(K, L) \), \( \dot{A}/A = g' \). This is called "Hicks neutral." One can also have capital augmenting technological progress. In the case of Cobb-Douglass production, they all come to the same thing. Specifically, if the production function is \( Y = K^{\alpha}/(eL)^{1-\alpha} \), then we can define \( A = e^{1-\alpha} \), and re-write the production function as \( Y = A K^{\alpha}L^{1-\alpha} \). Then starting with the definition of \( A \), taking logs, and differentiating with respect to time:

\[ \ddot{A}/A = (1-\alpha)\dot{e}/e \quad \text{So if } \alpha =.5, \text{ then labor augmenting technological progress at a rate of } 4\% \text{ per year is equivalent to Hicks neutral technological progress at a rate of } 2\% \text{ per year.} \]

Now we will apply a little trick, and everything will go through as before: we will define output and capital to be in per efficiency unit rather than per capita terms. That is: \( k = K/(eL) \) and \( y = Y/(eL) \).

Now we just take \((K/eL)\) and differentiate with respect to time:

\[
\frac{d}{dt}\left(\frac{K}{eL}\right) = \frac{eLK - KeL + K\dot{e}}{(eL)^2}
\]

\[= s\left(\frac{Y}{eL}\right) - \delta\left(\frac{K}{eL}\right) - n\left(\frac{K}{eL}\right) - g\left(\frac{K}{eL}\right) \]

So putting things back into per efficiency unit terms:
So now we have the same differential equation as before, and we can do all of the same analysis of steady states as before -- but there is one difference. What is "steady" in the steady state? The answer is that it is the amount of output per efficiency unit of labor -- not the amount per worker.

What is "steady" in the steady state? Answer: output per efficiency unit of labor.

What are the rates of growth of output per worker and of total output?

In math:
\[ y = \frac{Y}{eL} \]
\[ \ln(y) = \ln(Y) - \ln(e) - \ln(L) \]

\[ \frac{\dot{y}}{y} = \frac{\dot{Y}}{Y} - n - g \]

In the steady state, we know that this is zero, so \( Y / Y = n + g \), that is, total output grows at the sum of the rates of population growth and technological change.

Similarly, the growth rate of output per worker is

\[ \frac{d}{dt} \left( \frac{Y}{L} \right) = \frac{\dot{Y}L - \dot{L}Y}{L^2} = \frac{\dot{Y}}{Y} \cdot \frac{L}{L} = \frac{\dot{Y}}{Y} - n \]

and since \( Y / Y = n + g \), the growth rate of \( Y/L \) must be \( g \).

In words:

If output per efficiency units is constant, and efficiency units per worker is growing at rate \( g \), then it must be the case that output per worker is also growing at rate \( g \). Similarly if output per efficiency unit is constant and the total number of efficiency units is growing at rate \( n + g \), then total output must also be growing at \( n + g \).
The rest of what we know about the dynamics of the model still applies, but now in per-efficiency unit terms. Thus, for example, if a country starts off below its steady state in terms of output per efficiency unit, output will grow subsequently for two reasons: both because of movement toward the steady state, and because of growth in the number of efficiency units and in labor force. [picture with ln(Y/L)].

The Effect of a Change in Technology Growth

The analysis above says that an increase in g lowers k and y in the steady state. For example, using the Cobb-Douglas production function, steady state output is

\[ y_{ss} = \left( \frac{s}{n + g + \delta} \right)^{\frac{\alpha}{1 - \alpha}} \]

At first this result seems counter-intuitive, in that one would expect an increase in technological progress to raise the level of output. The resolution to this mystery is that faster technological progress does raise the level of output per worker, even as it lowers the level of output per effective worker, because it raises the number of effective workers per worker - that is, e. We can see this effect by tracing through the chain of events that follow from an increase in the rate of growth of technology, that is, in g0.

Earlier we derived the equation

\[ \dot{Y} = \dot{y} + n + g \]

which says that the growth rate of total output is the sum of the growth rate of output per effective worker, the growth rate of effective workers per actual worker, and growth of the labor force. In the steady state \( \dot{y} \) is zero. Now consider an economy that is in steady state. Suppose that there is an increase in the growth rate of technology - that is, g0 rises. There will be two forces acting on the growth rate of total output. On the one hand, g0 has risen. On the other hand, because the steady-state level of output per effective worker has fallen, \( \dot{y} \) will become negative (having been zero in the steady state). Which of these effects will dominate?

To answer this question, we rewrite the differential equation for capital stock by dividing both sides by k:

\[ \frac{\dot{k}}{k} = s k^{\alpha - 1} \cdot (n + g + \delta) \]
In the steady state, \( \dot{k} \) is zero. Let the increase in \( g \) be denoted \( \Delta g \). Because the right-hand side of the above equation is equal to zero before the increase in \( g \), we have that following the increase,

\[
\frac{\dot{k}}{k} = -\Delta g.
\]

The relationship between the growth rates of \( y \) and \( k \) can be derived by starting with the production function in per-effective-worker terms, taking logs, and differentiating with respect to time,

\[
\frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k}.
\]

Combining the two equations above,

\[
\frac{\dot{y}}{y} = -\alpha \Delta g.
\]

The growth rate of total output will thus be

\[
\frac{\dot{Y}}{Y} = \frac{\dot{y}}{y} + g + n + \Delta g = n + g + (1 - \alpha) \Delta g,
\]

so the initial effect of a rise in the growth rate of technology by some amount \( \Delta g \) will be to raise the growth rate of total output by \( (1 - \alpha) \Delta g \). Over time, however, as the economy moves to a new steady state, \( y \) will fall and \( \dot{y} \) will approach zero. In the new steady state, the growth rate of total output will have risen by the full amount \( \Delta g \).

The Solow Model for an Open Economy (improved, or at least extended version).

Consider an open economy characterized by the Solow model. We know that capital will flow into or out of the country to set the marginal product of capital, net of depreciation, equal to the world interest rate. That is:

\[
r = f'(k) - \delta
\]
Since some of the country's capital stock may be owned by foreigners (or some of the wealth of people in the country may consist of capital held abroad), we have to distinguish carefully between GNP and GDP. Let $k$ represent the quantity of capital in place in the country and let $a$ represent the quantity of assets owned by people in the country. Net foreign assets are $a-k$. In a closed economy, $k=a$.

We measure both GDP and GNP in per capita terms, so we put them in small letters. GDP is just output per capita from capital and labor in the country:

$$\text{gdp} = f(k)$$

GDP is the value of all output produced by factors owned by people in the country. We can alternatively look at this as just the value of all wage and interest payments to factors owned by people in the country:

$$\text{gnp} = (r+\delta)a + w$$

Note that the first term has $(r+\delta)$ multiplying the quantity of assets. The reason is that we want to calculate gross national product. The difference between gross national product and net national product is exactly that the latter adjusts for depreciation. Similarly, if we wanted to calculate net domestic product, that would be $f(k) - \delta k$. [Why are we looking at gross rather than net products? First, because that is what is measured in the data. Second, because when we did the original Solow model, we started with the assumption that saving was a constant fraction of gross domestic product, rather than a constant fraction of net domestic product.]

The second term is just $w$ (the per worker wage) times one (the number of workers per worker). To calculate the wage in the open economy, we just look at what is left after capital it paid its marginal product (note that the openness of capital markets implies that wages are the same everywhere, assuming that productive technology is the same everywhere):

$$w = f(k) - k f'(k) = f(k) - (r+\delta)k$$

Suppose that we have Cobb-Douglas production:

$$f(k) = k^\alpha$$

$$k = \left( \frac{\alpha}{r+\delta} \right)^{\frac{1}{\alpha}}$$
\[ w = (1 - \alpha)k^\alpha = (1 - \alpha) \left( \frac{\alpha}{r + \delta} \right)^{\frac{\alpha}{1 - \alpha}} \]

Once an economy is open, the level of capital jumps to the level determined by the interest rate. After that capital (per worker) is constant. Similarly, the wage is constant. There are still dynamics of the stock of domestically-owned assets, however. The differential equation governing their evolution is just

\[ \dot{a} = s[(r + \delta)a + w] - (n + g + \delta)a \]

So we can solve for the steady state level of domestic assets by setting \( \dot{a} = 0 \):

\[ a = \frac{ws}{n + g + \delta - s(r + \delta)} = \frac{s(1 - \alpha)}{n + g + \delta - s(r + \delta)} \left( \frac{\alpha}{r + \delta} \right)^{\frac{\alpha}{1 - \alpha}} \]

But it turns out that this steady state is not straightforward. We can graph the two pieces of the right hand side of the equation, putting \( a \) on the horizontal axis. There are two cases to consider. First, if \( s(r + \delta) < (n + g + \delta) \), then there is a conventional steady state. But if this inequality goes the other way, then there is a steady state, but it is unstable and negative [pictures]

which way should the inequality go empirically? Not clear.

Let's assume that the parameters are such that the steady state is stable.

Now let's think about the rest of the world. We will assume that it is described by a Solow model, with a constant saving rate \( s_w \) and the same production function as our country, and that the world is in steady state. Also, we will assume that our country is so small relative to the world that our saving rate has no effect on the world interest rate. Let \( k_w \) be the per capita level of capital in the world. We know that

\[ r + \delta = \alpha k_w^{\alpha-1} \]
and we can solve for the steady state level of world capital:

\[ k_w = \left( \frac{s_w}{n + g + \delta} \right)^{\frac{1}{1-\alpha}} \]

\[ r + \delta = \alpha \left( \frac{n + g + \delta}{s_w} \right) \]

Now we can solve for \( a/k \), that is, the ratio of domestic assets per capita to domestic capital per capita. Since \( k_w = k \) (since we have assumed that the domestic production function is the same as the world production function), this will be the same as \( a/k_w \). Dividing the equation for \( a \) by the equation for \( k_w \) and doing a lot of substituting (checked twice, but I'm too lazy to type in the algebra), we get:

\[ \frac{a}{k_w} = \frac{1-\alpha}{s_w - \alpha s} \]

This says that if we have the same saving rate as the world, the ratio of \( a/k \) is one. If we have a lower saving rate than the world, that \( a/k < 1 \).

We can now solve for the ratio of gnp to gdp:

\[ \frac{\text{gnp}}{\text{gdp}} = \frac{w + (\delta + r)a}{k^a} = \frac{(1-\alpha)k^a + (\alpha k^{a-1})a}{k^a} = (1-\alpha) + \frac{\alpha a}{k} \]

Does the free capital flow model work?
Feldstein and Horioka

Free trade as an alternative to factor flows: factor price equalization!
Discuss factor price equalization. One country case: take r and w as given, gives factor ratio in the two industries.

Now think about a small country that takes prices of two goods as given. This gives the unit value isoquants.

Now what is the unit value isoquant if we want to think about producing either good? It is the convex hull. This says that w/r will be given by the flat part, as long as both goods are being produced. In this case, quantities produced will be given by the arms of the trapezoid

We can also think about a world with just two countries and two factors and two goods (in trade, this is the 2x2x2 model.

first, we solve for the integrated equilibrium. This gives us w and r in the integrated world, and also gives us the quantities of each good produced and the factor ratios in each. We can draw a big trapezoid from these.

Then we look at the actual endowments. If the point for the actual endowment falls ....

Implications of the free trade model

One way of assessing the predictions of the free-trade model is by examining its implication for capital's share of national income, an issue that was first discussed in Chapter 3. If the free-trade model is correct, then GDP will be determined by the equation

\[ Y = rK + wL, \]

where the values of r and w will not change as capital is accumulated. (Since we are dealing with an economy that is closed to factor flows, we do not have to worry about the distinction between GDP and GNP). By contrast, in a closed economy, the accumulation of capital will lead to a reduction in r and an increase in w.

We can re-arrange the equation above to give:

\[ \text{Capital’s Share of Income} = \frac{rK}{rK + wL} = \frac{r}{r + w \left( \frac{L}{K} \right)} \]

This equation says that capital’s share of income will be higher, the higher is the capital/labor ratio. A quick examination of the data shows that this prediction does not hold true: there is no systematic
relationship between capital's share of income and the capital/labor ratio. This should not be a surprise, since the rough constancy of the capital/labor ratio was one of the pieces of evidence that was cited earlier in favor of the Cobb-Douglas production function.

Other implications of the free-trade model of growth are also contradicted when we look closely at the facts. For example, the model implies that rich countries (which have a lot of capital) should have exports that are heavily weighted toward capital intensive goods and imports which are heavily weighted toward labor-intensive goods. However, analyses of the "factor content of trade" do no show the sort of strong pattern that the model predicts.
Speed of convergence in the Solow model

Solow model in per-efficiency unit form:

\[ y = k^\alpha \]

Differential equation for capital:

\[ \dot{k} = sk^\alpha - (n + g + \delta)k \]

divide by \( k \) to get:

\[ \frac{\dot{k}}{k} = sk^{\alpha - 1} - (n + g + \delta) \]

We can graph the two parts of the right hand side of this equation.

[picture]

The difference between these two is \( \frac{\dot{k}}{k} \), the growth rate of the capital stock (per efficiency unit of labor). What about the growth rate of output? To see the relation between these, start with the production function, take logs and differentiate to get

\[ \frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k} \]

So the growth rate of output is just proportional to the growth rate of capital.

So what happens to the growth rate of output per capita away from the steady state? If we are below the steady state, the growth rate of output monotonically decreases as we approach the steady state.

What happens to the growth of output if we increase saving. in this case, the \( sk^{\alpha - 1} \) curve shifts up, we have a temporary increase in growth, then growth peters out to zero again.

We will now look more mathematically at the speed of output growth along the path to the steady state. We can re-write the production function as:

\[ \ln(y) = \alpha \ln(k) \]
define $y^*$ and $k^*$ as the steady state levels of income and capital per efficiency unit.

$$\ln(y^*) = \alpha \ln(k^*)$$

so

$$(1/\alpha)[\ln(y) - \ln(y^*)] = \ln(k) - \ln(k^*)$$

(note that we have left time index out for convenience)

We know qualitatively what will happen when $y$ is above or below its steady state level. But what can we say quantitatively?

To make progress, we will linearize around the steady state.

It turns out that to do this, we will want to re-write the equation for output growth in terms of the log of capital, and linearize in that variable: this is called "log-linearizing." So first we write output growth as a function of the log of capital:

$$\dot{y}/y = \alpha \ddot{k}/k = \alpha(s_k^{\alpha-1} - (n + g + \delta)) = \alpha(s e^{(\alpha-1)\ln(k)} - n + g + \delta) = f(\ln(k))$$

To linearize, we have to pick a point around which to do the linearization. We pick the steady state, that is, $\ln(k^*)$.

The linear approximation is:

$$\frac{\dot{y}}{y} \approx f(\ln(k^*)) + f'(\ln(k^*))\ln(k) - \ln(k^*)$$

the derivative of $f(\ln(k))$ is:

$$f' = \alpha s e^{(\alpha-1)\ln(k)}(\alpha - 1)$$
evaluating this at the steady state (when \( \dot{k}/k = 0 \)), we can replace \( s \times e^{(\alpha-1)\ln(k)} \) with \((n+g+\delta)\). So we get

\[
\frac{\dot{y}}{y} = \alpha(\alpha - 1)(n + g + \delta)[\ln(k) - \ln(k^*)]
\]

and we can replace the last part with the expression above to get:

\[
\frac{\dot{y}}{y} = (\alpha - 1)(n + g + \delta)[\ln(y) - \ln(y^*)] = \gamma[\ln(y^*) - \ln(y)]
\]

where \( \gamma = (1-\alpha)(n+g+\delta) \)

we can re-write this as:

\[
\frac{d\ln(y)}{dt} = \gamma[\ln(y^*) - \ln(y)]
\]

which says that the log difference between the current level of output per efficiency unit and the steady state level decays exponentially. Another way to write this is

---

To do this step we solve a differential equation as follows. For convenience, define \( x = \ln(y) \). So the differential equation becomes:

\[
\dot{x} = -\gamma(x - x^*)
\]

re-arrange:

\[
x + \gamma x = \gamma x^*
\]

multiply both sides by \( e^n \) and integrate:

\[
\int e^n \left[ \dot{x} + \gamma x \right] dt = \int e^n \gamma x^* dt
\]

\[
e^n x = e^n x^* + b
\]
\[
\ln(y_t) = e^{-\gamma t} \ln(y_0) + (1 - e^{-\gamma t}) \ln(y^*)
\]

If we want to know how long it takes the economy to close half of the gap in log income, we just set

\[\exp(-\gamma t) = 1/2\]

\[t = \ln(2) / \gamma \approx .7 / \gamma\]

so for \(\alpha=1/3, n=.01, g+\delta=.05\), this says that the "half life" of the log difference should be about 17 years. If \(\alpha\) is 2/3, it should be about 35 years.

(note that this is the half life of the log difference -- so it is like the time required to go from 1/4 of the steady state to 1/2 of the steady state). It is not the half life of the absolute difference.

Note that things like \(s\), which determine the level of the steady state, do not determine the speed at which the country approaches the steady state (this is true because of cobb-douglass production).

Extending the Solow Model to Include Human Capital

The dynamics in the Solow model come from the accumulation of capital. We have been thinking of capital as buildings and machines. The return to capital of this sort is interest and profits (In the real world, some of profit is also a return for "entrepreneurship," but we ignore this issue).

One of the facts that is consistent with the Cobb-Douglas production function is the constancy of capital's earnings as a share of output -- this is roughly 1/3.

where \(b\) is a constant of integration, which must be determined. We can re-arrange to get.

\[x = x^* + e^{-\gamma} b\]

Now we find the value of \(b\) from the starting condition: \(x(0) = x_0\).

\[x(0) = x_0 = x^* + b \Rightarrow b = x_0 - x^*\]

so

\[x = x^* + e^{-\gamma} (x_0 - x^*) = (1 - e^{-\gamma})x^* + e^{-\gamma} x_0\]
But now let's think about a broader definition of capital. Not just physical stuff, but also human capital. Invest and earn a return (higher wages) just like physical capital.

Most frequently run regression in economics: Human capital earnings function [graph].

What this says is that a large fraction of wages are returns to human capital rather than return to raw labor.

Thus, saying capital's share in income is 1/3 is wrong, since some of what we have been calling labor's share is really return to capital.

So let's think about a production function with two different kinds of capital, physical and human. For convenience we assume Cobb-Douglas:

\[ Y = K^\alpha H^\beta (e L)^{1-\alpha-\beta} \]

\[ \alpha + \beta < 1 \]

We will do our analysis in per-efficiency unit terms, so \( h = H/(eL) \), etc. So the production function is:

\[ y = k^\alpha h^\beta \]

We will assume that human capital and physical capital depreciate at the same rate, \( \delta \). Let \( s_k \) be the fraction of output that is used to produce physical capital, and \( s_h \) be the fraction of output that is used to produce human capital. The accumulation equations for physical and human capital are:

\[ \dot{k} = s_k k^\alpha h^\beta - (n+g+\delta)k \]

\[ \dot{h} = s_h k^\alpha h^\beta - (n+g+\delta)h \]

To find the steady state, we set both of these to zero and solve:

\[ k^* = \left( \frac{s_k^{1-\beta} s_h^{\beta}}{n + g + \delta} \right)^{\frac{1}{1-\alpha-\beta}} \]

\[ h^* = \left( \frac{s_k^{\alpha(1-\alpha)} s_h^{1-\alpha}}{n + g + \delta} \right)^{\frac{1}{1-\alpha-\beta}} \]
Now we can look at the speed of convergence to the steady state in this version of the model:

The equations for factor accumulation can be re-written as:

\[ \frac{\dot{k}}{k} = s_h k^{\alpha-1} h^{\beta} (n + g + \delta) \]
\[ \frac{\dot{h}}{h} = s_h k^{\alpha} h^{\beta-1} (n + g + \delta) \]

Starting with the production function, we can take logs and differentiate with respect to time to get:

\[ \frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k} + \beta \frac{\dot{h}}{h} \]

so we can write

\[ \frac{\dot{y}}{y} = \alpha s_h k^{\alpha-1} h^{\beta} + \beta s_h k^{\alpha} h^{\beta-1} (n + g + \delta) (\alpha + \beta) \]

\[ = f(\ln(k), \ln(h)) \]

The linear approximation for this case is
where \( f_1 \) indicates the derivative with respect to the first argument (note that the derivative is being taken w.r.t. \( \ln(k) \), not w.r.t. \( k \)).

\[
\begin{align*}
\dot{y} & \approx \alpha (\alpha + \beta - 1)(n + g + \delta)\ln(k) + \beta (\alpha + \beta - 1)(n + g + \delta)\ln(h) \\
& \approx -(1 - \alpha - \beta)(n + g + \delta)\ln(y) + \beta (\ln(h) - \ln(h^*))
\end{align*}
\]

We are evaluating this at the steady state, \( \frac{h}{h} = k/k = 0 \); so we can substitute from the equations of motion to get

\[
f \frac{\dot{k}}{\dot{h}}(\ln(k^*),\ln(h^*)) = \alpha(n + g + \delta)(\alpha - 1) + \beta(n + g + \delta)\alpha = \alpha(\alpha + \beta - 1)(n + g + \delta)
\]

and by symmetry we will have a similar thing for the derivative of \( f \) with respect to \( \ln(h) \). So putting these into the linearization formula (and noting that \( f(\ln(k^*),\ln(h^*)) = 0 \)), we get

\[
\frac{\dot{y}}{y} \approx \alpha(\alpha + \beta - 1)(n + g + \delta)\ln(k) - \ln(k^*)] + \beta (\alpha + \beta - 1)(n + g + \delta)\ln(h) - \ln(h^*)] \\
= (\alpha + \beta - 1)(n + g + \delta)\ln(k) - \ln(k^*)] + \beta (\ln(h) - \ln(h^*))
\]

but from the production function we have

\[
\ln(y) = \alpha \ln(k) + \beta \ln(h)
\]

so we can substitute for the term in brackets

\[
\frac{\dot{y}}{y} = -(1 - \alpha - \beta)(n + g + \delta)\ln(y) - \ln(y^*)
\]
[Students: Ignore this paragraph for now] We can compare these calculations of the speed of convergence in the extended Solow model to the results from MRW - specifically, they back out convergence speed from the coefficient on initial income in a growth regression -- get numbers around 1.5%.

**Capital Mobility in a Solow Model**

Now we do the model again, assuming that physical capital flows internationally to equalize the marginal produce less depreciation to some world rate r. r is exogenous. The assumption that whatever happens in our country has no effect on the world interest rate is usually called the assumption of a "small open economy"

First consider the Solow model with physical capital as the only factor of production.

Because the economy is open, we know that \( f'(k) - \delta = r \). This immediately pins down the level of the capital stock and the level of output. For example, for Cobb-Douglas production:

\[
    r = f'(k) - \delta = \alpha k^{\alpha-1} - \delta \\
    k = \left[\frac{\alpha}{r+\delta}\right]^{\frac{1}{1-\alpha}} 
\]

Notice that the saving rate doesn't matter at all to determining the level of output!

Distinction between GDP and GNP.....

Now consider the model with both physical and human capital. Obviously, if both human and physical capital flow between countries, then this will look just like the one-factor model just presented – in particular, the country will jump to its steady state immediately. Instead, we assume that while physical capital can flow between countries, human capital cannot. (The model is that of Barro, Mankiw, and Sala-i-Martin; except that they look at a fully optimizing model (like Ramsey), while I will do the "Solow" version).

\[
f_k - \delta = r \\
\]

from this, we can solve for the level of capital as a function of the level of human capital.

\[
\alpha k^{\alpha-1} h^\beta = r + \delta 
\]
\[ k = \left( \frac{\alpha}{r + \delta} \right)^{\frac{1}{\alpha}} \frac{\beta}{h^{\frac{1}{\alpha}}} \]

So we can solve for \( y \) as a function of just \( h \):

\[ y = h^{\beta} k^{\alpha} = h^{\frac{\beta}{\alpha}} \left( \frac{\alpha}{r + \delta} \right)^{\frac{\alpha}{r - \alpha}} \]

How does human capital accumulate? There is a potential problem here, since \( y \) is gross domestic product, but not gross national product (since capital may have flowed into or out of the country). One might think that human capital investment should be a constant fraction of GNP rather than GDP. On the other hand, we could argue that payments to human capital are a constant fraction of GDP (not GNP) because of cobb-douglass production. So if accumulation of \( h \) was financed by a constant fraction of payments to \( h \) (or payments to raw labor), then assuming it to be constant fraction of GDP would be OK. In any case, this is what we assume:

\[ \dot{h} = s_h y - (n + g + \delta) h \]

To get from here to the solution, there is a shortcut that we can take: the production function, once \( k \) is eliminated, looks just like the production function in the single factor case (where \( k \) was the factor) along with a constant in front. The only difference is that instead of the exponent being \( \alpha \), it is \( \beta / (1 - \alpha) \). We can just replace this for \( \alpha \) in the solution and immediately get the answer.

In the case where the production function was \( y = k^\alpha \), the solution was

\[ \frac{\dot{y}}{y} = (\alpha - 1)(n + g + \delta) \left( \frac{\ln(y) - \ln(y^*)}{\ln(y)} \right) \]

so substituting we get
\[
\frac{\dot{y}}{y} \approx \frac{(n + g + \delta)(\alpha + \beta - 1)}{1 - \alpha} \left( \ln(y) - \ln(y^*) \right)
\]

to be added (maybe) example of calculating actual speed of growth after a shock to saving rate, from homework problem.

So this is the full fledged Solow growth model. Question is: is it a good model?

Why good: gives testable implications about the effects of saving and population growth on output. (we shall see some tests later).

Why bad: not really a theory of growth in the sense that the main driving force of growth, technological progress, is takes as exogenous.

On the other hand, for a small country, maybe this is not a bad assumption. But for the world (and certainly for the US), it is clearly wrong.

There is something else missing in the Solow model: any explanation of how the rate of saving is determined. This is the question that we take up in the next model, the Cass-Ramsey model. (There is also no explanation of how population growth is determined, but that is not addressed in this course).

Cass-Ramsey model

The key difference between the Cass-Ramsey model and the Solow model is that, instead of taking the saving rate as exogenous, we will take it to be optimally determined. In fact, except for the determination of consumption, everything in the C-R model is exactly the same as in the Solow model.

Luckily, we have thought a lot already about the optimal determination of saving.

(we will do this first informally, then with a little more rigor).

For the moment, we will not think about markets, but will instead look at the problem of a social planner. [Who is the social planner: some benevolent guy who gets all of output and can decide what to do with it]. Later, we will take up the question of whether a de-centralized economy will
give the same result that a social planner would have achieved. (the answer will be yes for this model -- but not for some of the models we will look at later).

So we will have the same production function as in the Solow model, and writing everything in per worker terms, we have the same differential equation for the capital stock:

\[ \dot{k} = f(k) - c - nk \]

(where c is consumption and n is population growth. Note that there is no depreciation. If you want, you can say that f(k) is the net of depreciation production function).

We have to give the social planner a utility function:

What is going on here? U(c) is the instantaneous utility function, just like before. (Note it is really U(c(t)), but we suppress the time subscript for convenience). The business with e is the discounting, where \( \theta \) is in the same role (continuous time version) as it was in the discrete time version.\(^2\) The integral is just saying that the social planner's total utility is the sum of these things over the infinite future, in continuous rather than discrete time.

Note that what we are discounting is the utility of the average person in the next generation. That is, it is c that we are looking at, where c is per person consumption. You might think that what we really cared about was not only the consumption per person, but the total number of people who were doing the consumption. (reasons why or why not...) In any case, if we wanted to do this, it would be pretty simple. The total number of people per person today is just \( e^t \). So total utility would be

\[
V = \int_0^\infty U(c) e^{-\theta t} dt
\]

\[= \int_0^\infty U(c) e^{\theta t - nt} dt = \int_0^\infty U(c) e^{(n-\theta)t} dt\]

\(^2\)(this fn should really be in the consumption section)

\[
\left(\frac{1}{1+\theta}\right)^t = e^{\ln\left(\frac{1}{1+\theta}\right)} = e^{t \ln\left(\frac{1}{1+\theta}\right)} \approx e^{-\theta t}
\]

Where the last step is true because \( 1/(1+\theta) \approx 1 - \theta \) and \( \ln(1-\theta) \approx -\theta \); both these approximations are close to exact for \( \Theta < 0.1 \).
Note that if $n > \Theta$, then total utility would be infinite (which is a problem that can be dealt with, but it is a pain). So let's just consider cases where this is not true.

So that is all there is to the setup.

One thing that we have to say to make this more comparable to the individual's consumption optimization problem that we were looking at before is what is the interest rate as far as the social planner is concerned? To see the answer: consider what he gets if he puts aside a piece of capital for one period. (remember that we are ignoring depreciation here). Well, in the next period, he will get back $(1+\text{mpk})$ units of capital. But there will be more people: each unit of capital per capita will have to be divided among $(1+n)$ people. So the total amount will be $\frac{(1+f'(k))}{(1+n)}$ which is approximately equal to $(1 + f'(k) - n)$. So as far as the social planner is concerned, $(f'(k) - n)$ is the interest rate that he is facing in making his optimal consumption path decision.

So now let's consider in loose terms what is going to be the case with the time path of consumption.

From the discrete time optimal consumption problem we have some things that we know:

if $\Theta > r$ then consumption is falling, etc. So in this case, the translation is just, if $\Theta > (f'(k)-n)$, then consumption must be falling. Similarly, if they are equal, then consumption is constant, and if $(f'(k)-n) > \Theta$, then consumption is rising. (all this must be true along the optimal path.).

In fact, in section 1 of the course, we showed that if utility is CRRA and the interest rate is constant, then the growth rate of consumption along the optimal path (in continuous time) was

$$\frac{\dot{c}}{c} = \frac{I}{\sigma}(r - \Theta)$$

So now we draw a set of axes for $c$ and $k$. For our own reference, let's also write in what is happening to the MPK as we move along the $k$ axis. When $k$ is low, MPK is high. When $k$ is high, MPK is low. At some point in the middle, MPK is equal to $\Theta$.

Now draw in the $\dot{c} = 0$ locus. This is a vertical line drawn over the point on the $k$ axis where $f'(k) = \Theta + n$. Then draw in the arrows for $c$. Explain carefully.

Now forget about consumption for a moment. Let's go back to the capital stock differential equation. Suppose we wanted to draw in the locus of all of the points where $\dot{k}$ is equal to zero.

the capital stock accumulation equation is:
\[ \dot{k} = f(k) - c - nk \] (remember, depreciation is zero)

so \[ \dot{k} = 0 \] locus is just

\[ c = f(k) - nk \]

What does this look like? Well we can just graph the two components on the right side, like we did before -- then c is the difference between them.

This is the locus of all of the points where consumption is sustainable (since if \( \dot{k} = 0 \), then it is a steady state.) We can see that the golden rule capital stock is just at the peak of this curve -- that is where it's derivative is zero, and so \( f'(k) = n \).

Now, the big question: What is the relationship between the \( \dot{c} = 0 \) locus and the top of the \( \dot{k} = 0 \) locus?

Since \( f'(k) = \Theta + n \) for the \( \dot{c} = 0 \) locus, and \( f'(k) = n \) at the top of the \( \dot{k} = 0 \) locus, and since the marginal product of capital falls as k rises, the \( \dot{c} = 0 \) locus must lie to the left of the top of the \( \dot{k} = 0 \) locus.

Now draw in the \( \dot{k} \) arrows: if we are at a point underneath the \( \dot{k} = 0 \) locus, k will be rising over time. If we are above the locus, k will be falling.

Talk about how the arrows tell you where you would go given any initial position.

Now draw in steady state and the stable arm.

What happens if you are not on the stable arm...

Note that the stable arm is really incorporating your budget constraint.

So wherever k starts, you jump to the stable arm.

Note that the steady state is never above the golden rule. In fact this steady state, at which \( f'(k) = n + \Theta \), is called the "modified golden rule." Why do you end up here, rather than at the golden rule which, we already established, is the best steady state at which to be? That is, golden rule maximizes your steady state consumption.

Answer: discounting. If \( \Theta = 0 \), then do end up at golden rule. It is always feasible to get to the golden rule, but not worth it. [time series picture]

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Effects of parameters of the phase diagram:
Increase in $\Theta$: shifts in the $\dot{c} = 0$ locus

increase in $\sigma$: rotates the stable arm to be closer to the $\dot{k} = 0$ locus. (how to derive this: raising $\sigma$ means that at any given point in the phase space, the movement in the vertical direction will be slower than when $\sigma$ is small. Thus starting on the old stable arm will lead to missing the steady state, etc.).

increase in $n$: shifts $\dot{k} = 0$ locus down and $\dot{c} = 0$ locus to the left. Depending on the parameters, the new stable arm can be above or below the old steady state.

Adding Government to the model

We will assume that there is some spending $G$ (per capita -- but we use a capital $G$ to differentiate it from the growth rate).

Note that we do not have to say whether this spending is financed by taxes or borrowing -- why? Because the assumptions of Ricardian Equivalence hold (no borrowing constraints, forward looking behavior).

Government spending does not affect production or utility. It only goes into the capital accumulation equation, which is now:

$$\dot{k} = f(k) - c - G - nk$$

So a shift up in $G$ lowers the $\dot{k} = 0$ locus

Suppose that we are at steady state, and $G$ increases... this is easy. If we were not at steady state, we would jump down from the old stable arm to the new one.

Suppose that we were at steady state, and it was announced that $G$ was going to rise some time in the future? This is an anticipated change. We know from section 1 that there can't be an anticipated jump in consumption. So when the change arrives, we must be on the new stable arm. So consumption must jump instantaneously. The size of the jump will depend on the time until the new level of spending: if it is far away, then there will only be a small jump (since dynamics around the steady state are very small).

Now suppose that there is a temporary (but immediate) increase in $G$. We know that when the $G$ moves back to its initial (and final) level, we must be on the stable arm. In the interim, we move according to the dynamics associated with the higher level of $G$.

We can go on: think about an anticipated temporary increase in $G$....
We can also look at how interest rates behave (see Romer book). We see that a temporary increase in \( G \) raises interest rates, while a permanent increase doesn't. Why? For a temporary increase, people have to be induced to temporarily reduce consumption -- that is, to have an upward sloping path. Only way to do this is via higher \( r \).

Aside: note that these experiments in which we "shock" the economy -- i.e. you wake up one day and \( G \) is higher -- really are trying to deal with uncertainty in a certainty model. That is, we assume that when people make their consumption decisions they are operating with certainty. But then we say that things change -- if such a thing were possible, then people should have made their decisions assuming that changes were possible -- that is, that the world was uncertain.

Doing the Hamiltonian.

We now do the whole thing more formally, using the Hamiltonian. This method is also known as the Maximum Principle and Pontryagin's method. This is going to look like plug and chug for now, but maybe it will be useful someday.

The problem:

\[
\max \int_0^\infty U(c)e^{-\theta t} dt
\]

s.t.

\[
\dot{k} = f(k) - c - nk
\]

\[k(0) = k_0\]

where we should note that \( c \) and \( k \) both vary over time -- that is, we could write them as \( c(t) \) and \( k(t) \).

So now we open our books, and see that the Hamiltonian can be used to solve the general problem.
The control variable is the thing that you get to move around, and the state variable is the thing that tells you what state of the world you inherited from the last instant. Note that the state variables never change discontinuously, while the control variable can do so. The second equation is the equation of motion for the state variable, and the third equation just says that you are given some initial value of the state variable. Note that in bigger problems you can have several state or control variables. In such a case, there will be an equation of motion and an initial value for each state variable. $I(\cdot)$ is the objective function.

For example, in the problem of landing a rocket on the moon, your state variables will be your speed, your distance from the moon, and the amount of fuel you have -- and your control variable will be the amount of fuel that you burn each instant.

The book then says to write down the hamiltonian as

$$H = I(x,u,t) + \lambda x f(x,u,t)$$

where lambda is called the costate variable. Note that this looks a lot like a Lagrangian -- except that now, instead of having a single value, $\lambda$ will be allowed to vary over time -- that is it is really $\lambda(t)$.

Sometime later in the math course, you will learn that this is just like the lagrange multiplier technique.

We open up a book (I use Intriligator, chapter 14) and see that the first order conditions for a maximum are:

$$\text{Max } \int_{t_0}^{t_f} I(x,u,t) dt$$

s.t.

$$\dot{x} = f(x,u,t)$$

and

$$x(0) = x_0$$
\[ \frac{\partial H}{\partial u} = 0 \]

(where \( u \) is the control variable)

and

\[ \dot{\lambda} = -\frac{\partial H}{\partial X} \]

(where \( x \) is the state variable)

[technical note: we also have to incorporate any initial and/or terminal conditions....]

So this is what we want to apply to our problem. The state variable is \( k \), and the control variable is \( c \).

We write down our hamiltonian:

\[ H = U(c)e^{-\theta t} + \lambda (f(k) - c - nk) \]

[Note that the general formulation of the Hamiltonian allows for things that we don't care about. For example, it allows for the objective function to include values of the state variable.]

Applying the first of the first order conditions to the problem at hand we get that

\[ \frac{\partial H}{\partial c} = U'(c)e^{-\theta t} - \dot{\lambda} = 0 \]

or

\[ U'(c)e^{-\theta t} = \dot{\lambda} \quad (1) \]

and for the second, we get

\[ \dot{\lambda} = -\lambda (f \dot{k} - n) \quad (2) \]
So these are the two equations that hold at all times along the optimal path. That is, at every instant there is a different value of $c$, $k$, and $\lambda$, but these two conditions must always hold.

The first of these conditions is a relation between the $c$ and $\lambda$, while the second tells us about how $\lambda$ changes over time.

But note that the one that tells us about how $c$ and $\lambda$ relate can also tell us about how $\theta$ and $\lambda$ relate. That is, we can take the first condition and differentiate with respect to time to get (by the product rule):

$$\dot{\lambda} = -\theta e^{\theta U'(c)} + \dot{c} e^{\theta U''(c)}$$  \hspace{1cm} (3)

Combining (1) and (3) gives:

$$\frac{\dot{\lambda}}{\lambda} = -\theta + \dot{c} \left( \frac{U'(c)}{U'(c)} \right)$$  \hspace{1cm} (4)

(note that all we did in manipulating (1) to get to (4) was to take a relationship between $c$ and $\lambda$ that had to hold true in levels, and convert it to one expressed in rates of growth.

We can also re-arrange (2) into growth-rate form to get:

$$\frac{\dot{\lambda}}{\lambda} = \frac{-f(k) + n}{\dot{c}}$$  \hspace{1cm} (5)

Finally, we combine (4) and (5) and re-arrange to get

$$\dot{c} = -\left( \frac{U'(c)}{U''(c)} \right) [f(k) - (n + \theta)]$$

So now we are done ... sort of. We have an equation that is basically the same as what we were looking at the intuitive level, which tells us how consumption should change, given a particular level of consumption and of the capital stock.

If you go and read Intriligator or something else, you will see that there are other constraints on $\theta$, $\lambda$, and $\lambda'$ that has to impose to make sure that the path does not go off to infinity somewhere -- but these are
basically the same as saying that you have to be on a stable arm that goes into the steady state.

You also might ask what has happened to the curvature of the utility function (aka risk aversion). This is incorporated into the $u'(c) / u''(c)$ term.

To be concrete, use CRRA utility function:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma} \quad \sigma > 0 \quad \sigma \neq 1$$

$$= \log(c) \quad \text{if } \sigma = 1$$

We now calculate $u'(c) = c^{-\sigma}$

and $u''(c) = -\sigma c^{-\sigma-1}$

We can plug these into the equation to get:

$$\frac{\dot{c}}{c} = \frac{1}{\sigma} (f'(k) - (n + \theta))$$

This looks a whole lot like the first order condition for consumption growth that we derived for the CRRA utility function when we looked at consumption in discrete time. Just as in that case, the extent to which consumption responds to the stuff in parenthesis depends on $\sigma$. If sigma is near zero, then consumption responds a lot. etc.

Discuss transition paths in response to shocks as fns of $\sigma$.

The Decentralized economy

Now we examine the question of whether the paths of $k$ and $c$ chosen by the social planner will be the same as the paths chosen by a decentralized economy. By a decentralized economy, we mean one in which there are a lot of agents (but each is still infinitely lived, so we call them families). Each agent takes a path of $w$, the wage rate, and $r$, the interest rate, as given. They then choose an amount of consumption on the basis of these. So the family's problem is

$$\text{Max} \quad \int_{0}^{\infty} e^{-\delta t} U(c) \, dt$$
s.t.

\[ \dot{a} = w - c + (r - n)a \]

Where \( a \) is the amount of assets that the family holds, and \( a_0 \) is given.

So we set up the family's hamiltonian

\[ H = e^{-\Theta t} u(c) + \lambda (w - c + (r - n)a) \]

and we get the first order conditions:

\[ \frac{\partial H}{\partial c} = 0 \quad \Rightarrow \quad \lambda = U'(c)e^{-\Theta t} \]

\[ \frac{\partial H}{\partial a} = \dot{\lambda} = -\lambda(r - n) \]

If we do the same manipulations that we did earlier, we get the equation of motion as

\[ \dot{c} = -\frac{u'(c)}{u''(c)} (r - n - \Theta) \]

and the equation of motion for assets is just (from above)

\[ \dot{a} = w - c + (r - n)a \]

But now note a few things. First, we can write \( w \) and \( r \) as functions of \( k \):

\( r = f'(k) \)[the interest rate is the MPK less depreciation; but we said depreciation was zero]

\( w = f(k) - k f'(k) \)  [wage is what is left over -- production is CRS, so factor payments exhaust output].

Finally, there is the relation between \( a \) and \( k \). Well, clearly, since this is a closed economy, there is no other asset that people can hold except capital. So it has to be the case that \( a = k \).

Substituting these three things into the two equations of motion gives:
\[
\dot{c} = - \frac{u'(c)}{u''(c)} (f'(k) - n - \theta)
\]

\[
\dot{k} = f(k) - f'(k)k - c + (f'(k) - n)k = f(k) - c - nk
\]

Which are, of course, the same equations of motion that we had before.\(^3\)

So: imagine that we have a bunch of atomistic households, each with initial wealth per capita of \(a_0\). Each household takes the future paths of \(r\) and \(w\) as given, and then chooses its own path of consumption and of wealth accumulation. What we have shown is that the same path of capital and consumption chosen by the social planner would be an equilibrium path for this set of atomistic households -- in the sense that if the households expected this path of \(w\) and \(r\), they would act in such a way that capital would grow over time at a rate that would produce the expected paths of \(w\) and \(r\). So rather than think about how the different households interact, we can just solve the social planner's problem. Furthermore, this is one of those cases where the decentralized outcome is the same as the social planner's choice -- showing that markets are a good thing.

→ talk though cases where decentralized does not match social planner’s solution.

[End of Hamiltonian]

Summary: So this is the Ramsey model. How does it differ from Solow? In the steady state, the two are the same!!!! But now saving is being taken as endogenous. So we can say how the rate of time discount determines the saving rate and thus the steady state. (note that the other parameter of preferences -- the curvature of the utility function -- says something about the transition but not about the level of the ss).

[problem for here, homework, or test: what can you say about the rate of saving along the transition path? Consider the effects of the coefficient of relative risk aversion. Explain intuitively and mathematically. Can you solve for the exact path? (note: I can't so far).]

The OLG Model

\(^3\) In the background here are the transversality conditions: We have to impose condition on the family's assets: that they can't keep borrowing indefinitely. -- this turns out to be the same as the condition that the capital stock does not go to zero or become infinite; in other words that it is on the stable arm.
We now look at a different growth model that is useful in a large number of contexts. Here we will use a very simple version, so that you can see how it works and see how a change in the structure of a model can lead to different conclusions.

OLG -- the idea.

The idea behind this model is to deal explicitly with the idea that people do not live forever, and that different people are at different stages in the life cycles at different times. If you look around in the world, you can see that some people do not own any capital, but are saving up capital from the wages they earn working, while other people have stopped working, and are now living off their capital. This is, of course, the Life Cycle model of saving. But here we try to model it in a stripped down form.

So the setup of the model is as follows:

The model takes place in discrete time. Note that discrete vs continuous time is not a key difference between the Ramsey and OLG models. One can write down a continuous time OLG or a discrete time Ramsey model. The key difference is the infinite horizon of Ramsey consumers vs the life cycle horizon of OLG consumers.

People live for two periods. They come into the world with no assets. In the first period, they work and consume some of their wages. In the second period, they do not work, but earn interest on the capital. They consume the capital and their interest, then die.

Within each period, production takes place first, then factors get paid, then consumption takes place.

We will again do everything divided by the number of people in the generation

So if $s_t$ is the savings of people in the young generation

$$s_t = w_t - c_{1,t}$$

where $w_t$ is the wage of a person who works in period $t$ and $c_{1,t}$ is the consumption of a person who is young in period $t$.

Similarly, the consumption of a person who is old in period $t+1$ is

$$c_{2,t+1} = (1+r_{t+1}) s_t$$

where $r_{t+1}$ is the interest rate earned in period $t+1$, and $s_t$ is the amount saved by the people who were young in period $t$, who are old in period $t+1$.

Note that we can combine these two equations to get the consumer's budget constraint:

$$w_t = c_{1,t} + c_{2,t+1}/(1+r_{t+1})$$
We allow for population growth by making each generation \((1+n)\) times as big as the one before it: \(L_{t+1} = (1+n) L_t\)

Thus the per worker capital stock in period \(t\), \(k_t\) is equal to the savings of the people who are currently old, adjusted for the fact that the number of workers is greater than the number of old people by the factor \((1+n)\)

\[
k_{t+1} = \frac{s_t}{1+n}
\]

Now we have to specify how the worker decides how much of his earnings to consume in the first period, and how much to save for the second period. We give him the usual utility function, with discount rate \(\Theta\).

So his problem is

\[
\text{Max } U = u(c_{1,t}) + \frac{u(c_{2,t+1})}{1+\Theta}
\]

s.t. \(w_t = c_{1,t} + c_{2,t+1}/(1+r_{t+1})\) \(\text{(the budget constraint)}\)

You can solve this by doing a lagrangian to get the first order conditions:

\[
L = u(c_{1,t}) + u(c_{2,t+1})/(1+\Theta) + \lambda \left[ w_t - c_{1,t} - c_{2,t+1}/(1+r_{t+1}) \right]
\]

\[
dL/dc_1 = 0 \implies u'(c_{1,t}) = \lambda
\]

\[
dL/dc_2 = 0 \implies u'(c_{2,t+1})/(1+\Theta) = \lambda (1/(1+r_{t+1}))
\]

combining these gives:

\[
u'(c_{2,t+1}) / u'(c_{1,t}) = (1+\Theta)/(1+r_{t+1})
\]

Which should look familiar

To make our life easier, we will now use a specific form of the utility function which turns out to be convenient in a lot of circumstances:

\[
u(c) = \ln(c) \quad \text{so } u'(c) = 1/c
\]
putting this specific utility function into our FOC gives:

\[
c_2 / c_1 = (1+r)/(1+\Theta)
\]

combining this condition with the budget constraint gives explicit expressions for consumption (get \(c_2\) alone in each equation):

\[
c_2 = c_1 (1+r)/(1+\Theta)
\]

\[
c_2 = (w - c_1)(1+r)
\]

\[
=> \quad (w-c_1) = c_1 / (1+\Theta)
\]

\[
=> \quad c_{1,t} = ( (1+\Theta)/(2+\Theta) ) \ w_t
\]

so saving is:

\[
s_t = (1 / (2+\Theta)) \ w_t
\]

Note that the expression for \(c_1\) and \(s_1\) is not affected by the interest rate! This is not a general result -- rather it is peculiar to log utility.

[Aside: where does this result about log utility come from?

Consider the problem:

\[
\begin{align*}
\text{max} & \quad \alpha \ln(c_1) + \beta \ln(c_2) \\
\text{st.} & \quad w = p_1 \ c_1 + p_2 \ c_2
\end{align*}
\]

where the \(p\)’s are some prices and \(w\) is the endowment.

The solution is

\[
\begin{align*}
c_1 &= (w/p_1)(\alpha / (\alpha + \beta)) \\
c_2 &= (w/p_2)(\beta / (\alpha + \beta))
\end{align*}
\]

this says that the fraction of the endowment spent on \(c_1\) (that is \(c_1 p_1 / w\)) is just equal to \(\alpha / \alpha + \beta\) -- and that it doesn't depend on prices!

What is a change in the interest rate in an intertemporal consumption problem other than a change in price? (that is, the price of a unit of consumption in period 2 is \((1/(1+r))\). Therefore, with log utility, changes in the interest rate do not affect the total amount of the endowment set aside for consumption in each period, and does not affect period 1 consumption. [note that if some income were being earned in period 2, the result that period one consumption was invariant to the interest rate would not hold, since a change in the interest rate affects the p.d.v. of lifetime income].

Here the problem has been done for intertemporal consumption, but it could just as easily have been done for choice of consumption among two goods -- that the total fraction of wealth spent on each good does not depend on price (put another way, demand has an elasticity of one).
Note finally, that the log utility function can be translated into our old friend the Cobb-Douglas utility function:
\[ \alpha \ln(c_1) + \beta \ln(c_2) = \ln(c_1^\alpha c_2^\beta) \]
and whatever maximizes \( \ln(c_1^\alpha c_2^\beta) \) must also maximize \( c_1^\alpha c_2^\beta \). So our result is really the same as saying that with a Cobb-Douglas utility function you spend some amount \( \alpha/(\alpha+\beta) \) on the first good no matter what the price is. And this result is the same as the one in production that says that, for CRS and Cobb-Douglas production, each factor's share of total output is given by its exponent.

end of aside.]

So now we have to say where \( r \), the interest rate, and \( w \), the wage rate come from. That is, we have to specify the production function.

Again, we will take the easiest, off-the-shelf production function:

\[ f(k) = k^\alpha \]

so the fraction of output that goes to capital is \( \alpha \), and the fraction that goes to labor is \( (1-\alpha) \). So the wage is just

\[ w_t = (1-\alpha) k_t^\alpha \]

The interest rate that people earn on their savings is the marginal product of capital, less depreciation.

\[ r_t = f'(k_t) - \delta = \alpha k_t^{\alpha-1} - \delta \]

Aside:

If we had not assumed log utility, then we would have to solve a system of equations
\[ s_t = s(w_t, r_{t+1}) \]
\[ r_{t+1} = f'(k_{t+1}) \]
\[ k_{t+1} = s_t / (1+n) \]

so that we could derive the function
\[ s_t = g(w_t) \]

and then work with that to get the dynamics of \( k \)

[end of aside]
We can combine all of these into an equation that gives $k_{t+1}$ as a function of $k_t$. This is a first order difference equation.

$$k_{t+1} = \frac{(1-\alpha)/[(1+n)(2+\Theta)]}{(1-\alpha)}k_t$$

We can draw a simple picture to look at this function, with $k_t$ on the x axis, and $k_{t+1}$ on the y axis. We graph the function as well as the 45-degree line. The interpretation of the place where the function crosses the 45-degree line is easy: if $k$ is at this level in one period, it will be there for the next, and indefinitely thereafter.

Note also the dynamics -- if you start off below the steady state capital stock, then in period $t+1$, capital is greater. So this equilibrium is stable.

For the functions that we are using, it is clear that there is only one crossing point. But this is not necessarily the case. You can have utility functions and production functions such that there are several different crossing points. Each of these will be an equilibrium. But note that some of these equilibria will be stable, while others will be unstable. [pictures]

But for the case that we are dealing with, this is not a problem.

We can solve for the steady state by setting $k_t = k_{t+1}$.

$$k^* = \left(\frac{1-\alpha}{(1+n)(2+\Theta)}\right)^{1/(1-\alpha)}$$

we can also solve for the interest rate

$$f'(k) = \alpha k^{\alpha-1} = \alpha (1+n)(2+\Theta)/(1-\alpha)$$

$$r = [\alpha (1+n)(2+\Theta)/(1-\alpha)] - \delta$$

Now let's look at the question of the golden rule. The golden rule is defined as

$$f'(k_{gr}) = n + \delta$$

or alternatively, at the golden rule $r = n$
if the capital stock is less than the golden rule, then r>n, and if capital stock is greater than the golden rule, then n>r.

Two points to notice:

1) There is no reason given the parameters that r has to be bigger or smaller than n. So that we can easily be above the golden rule capital stock. This is clearly inefficient, since all current and future generations could be made better off by lowering the capital stock (ie lowering the saving rate).

2) Clearly being above the golden rule is bad, since if we just saved less, then there would be more output in every period. As a dramatic way to see this, think about how you could make everyone better off by just implementing the following policy: every young person gives every old person a dollar. So each person gives away 1 dollar, but gets back (1+n) dollars -- this is better than getting (1+r) dollars that the person would have gotten by putting his money into capital.

This state of the world, in which saving is too high, and in which everyone can be made better off -- that is, there can be a Pareto improving re-arrangement -- is called dynamic inefficiency. Note that we showed that this could not happen in the Ramsey model. Essentially, what is going on is that in the Ramsey model, there was a sense in which a single person (either the social planner or individuals) was optimizing over all of the different time periods in the future -- so clearly if there was an improvement to be made by moving stuff from one period to another, they would do it. In the OLG model, no one person is present in all periods to do this trading, and so it is possible to have dynamic inefficiency.

Note the sorts of conditions that can give you dynamic inefficiency: having a small (or negative) value of \( \Theta \) will lower r, and make it more likely that we have dynamic inefficiency; similarly having a low value of \( \alpha \) will make it more likely that there is dynamic inefficiency. In both these cases, people are trying to shift too much consumption to the later part of life, driving down the marginal product of capital.

Another note about dynamic inefficiency, consider the possibility of bubbles in this economy. Suppose that you have some intrinsically value-less object. We want to ask the question: is it possible that everyone in the economy can agree that this is valuable, and pass it on from generation to generation. Well, at what rate must the value of the asset grow? Answer: r. If not, then people would not hold it. So what happens if r>n? Then the total value of the asset is growing faster than the economy, and eventually there is more of the asset than people want to hold, and so people will not buy it. So this cannot be an equilibrium. On the other hand, if n>r, then there is no problem.

As an empirical matter, we have probably never seen dynamic inefficiency.

Effect of social security
We now use the OLG model to look at the effect of a real life policy. Social security is a program that takes money away from you when you are young, and gives it to you when you are old. [In theory there are two types of SS systems: funded, and pay-as-you-go. In reality, only the latter has ever been observed.]

So call the tax that you pay when you are young, \( t \)

the benefit that you get when old is \((1+n)t\)

[We can add growth as follows. Suppose that the amount paid by every young person grows at rate \( g \) per generation (because wages are also growing at rate \( g \) and the ratio of payments by the young to their wages is constant. Then the benefit per old person at time \( t \) will be \((1+n)t\), which is the same as before, but expressed in terms of the taxes \( they \) paid when young it will be \((1+n)(1+g)t\].

Note that it is bigger than what you paid, because there are more young people than old people. [Note that in the early days of SS, it was easy to have high benefits, because there were a lot more young people than old people; but now that population growth has slowed down, these high benefits are more costly to the young.].

What happens to your budget constraints?

first period: \( s_t = w_t - c_{1,t} - t \)

second period: \( c_{2,t+1} = (1+r_{t+1})s_t + (1+n)t \)

We can combine these two equations to get the consumer’s budget constraint:

\[
W_t + \frac{(n-r)}{(1+r)}t = c_{1,t} + \frac{c_{2,t+1}}{1+r}
\]

Note that if \( n=r \), then social security has no effect on the consumer’s budget constraint. If \( n>r \), then social security has the effect of expanding the amount that the consumer can consume over the course of his lifetime. If \( r>n \), then SS has the net effect of making the consumer worse off. So why might we have SS in this case? Answer: in the real world, there are other considerations. For example, maybe people aren’t so forward looking, and so don’t provide for themselves. (also the first generation to receive SS got it for free).
If we plug our new budget constraint into a lagrangian we can solve for optimal consumption and saving. We get for consumption:

$$c_{t+1} = \frac{1 + \theta}{2 + \theta} \left[ w_{t+1} \left( n - r_{t+1} \right) d + \frac{(n - r_{t+1})d}{(1 + r_{t+1})} \right]$$

and for saving (after several intermediate steps):

$$s_{t} = \left( \frac{1}{2 + \theta} \right) \left[ w_{t} \left( n - r \right) (1 + \theta) d - d \right]$$

And if we combine this new saving rate with the equations for the capital stock, the wage rate, and the interest rate, we can get an expression for $k_{t+1} = g(k_t)$ [note that the expression is a mess, because the $r$ that you have to use is a function of $k_{t+1}$, so you have $k_{t+1}$ on both sides of the expression, and may not be able to solve explicitly for the function. Still, you can convince yourself that the new $k_{t+1} = g(k_t)$ function lies below the old $k_{t+1} = g(k_t)$ function.

So we get:

[picture]

So what happens to the steady state capital stock? -- it has to go down. If you are above the golden rule, this is an unambiguously good thing. If you are below the golden rule, it is more complicated. [why? : because we haven't set up an objective function that would allow us to decide how to weight the welfare of different generations. ...]

A note on making this conform to the real world: obviously, as we have written it down, the OLG model is not really very realistic. But one can do a much better job. If you look in Auerbach and Kotlikoff, you can see several of the things that one can do -- if one is willing to give up solving analytically and instead solve numerically on a computer -- to make the model more realistic. For a start, the divide the lifetime into 55 periods -- from age 20 to age 75. They also do not enforce a mandatory retirement age (ie, in the two period OLG model you have to retire halfway through your life) -- instead, they make it so that as you get older, you become less productive, and so you optimally chose to stop working. They also incorporate things like taxes and the social security system into their model. They then choose some realistic functions and parameters for the utility function and the production function. Then they take the model and use it to do realistic
policy experiments -- e.g. what if we change the rules of the social security system, or change from taxing wages to taxing consumption. And they do not just solve for steady states, but also for the complete transition paths -- that is, each person decides what is optimal for him to do today, given the next 55 years of interest and wage rates -- and given what members of each generation decide to do, the capital stocks and labor supplies are determined, and these determine the interest and wage rates -- so the computer has to find a fixed point path of these things -- which is a big mess.

Of course, the problem with using these simulations for policy analysis is that there may be other reasons for saving besides the ones highlighted in the OLG model: precautionary savings, liquidity constraints, bequest motives, etc.

Open Economy

Optimal consumption (and then growth) for an open economy. (Follows Obstfeld and Rogoff, Ch 2, very closely).

Define $B_t$ as net foreign assets at time $t$.

We take the interest rate as exogenous and constant at $r$ -- this is the "small open economy" assumption. Nothing that happens in our country will affect the world interest rate.

The Current account is the change in net foreign assets. (This is the same as the trade surplus in the case where we are not holding any foreign assets to begin with. More generally, it is equal to the trade surplus plus interest on the assets we hold abroad, minus interest on the debt that we owe foreigners.).

CA is defined as:

$$CA_t = B_{t+1} - B_t = rB_t + Y_t - C_t - G_t - I_t$$

We can define the trade balance (same as net exports) as

$$TB = Y - C - I - G$$

so we can re-write the definition of the current account to say:

$$B_{t+1} - B_t = rB_t - TB_t$$

This says that to hold the foreign debt constant (ie to have a zero current account) we have to set our trade balance equal to the interest on the debt.
Rather than holding its foreign debt constant, a country might be interested in keeping the ratio of debt to income constant. Suppose that income grows exogenously at rate $g$. To hold $B/Y$ constant, $B$ must also grow at rate $g$.

So

$$B_{s+1} - B_s = gB_s = rB_s + TB_s$$

re-arranging

$$TB_s = -(r-g)B_s$$

dividing both by $Y_s$

$$\frac{TB_s}{Y_s} = -(r-g)\frac{B_s}{Y_s}$$

So to maintain a constant debt/income ratio, you have to maintain a constant ratio of trade balance to GDP, but the size of that ratio depends on the difference between $r$ and $g$. If they are close, then the trade balance to GDP ratio can be quite small.

[note: we have been assuming that $r>g$. Why? If this were not the case, then one could never pay any interest on ones debt, and still the debt/income ratio would fall over time. In this case, any model of borrowing in which one tried to keep the debt ratio constant would not make sense. As a practical matter, although some countries have period of time in which $g$ is higher than $r$ (when they are transitioning to a higher steady state), it seems clear for countries in SS that $g<r$.]

We can substitute the differenetial equation for the evolution of $B$ forward, just like we did the equation for assets of an infinitely-lived consumer. For the infinite horizon case (incorporating implicitly a transversality (or no-ponzi-game condition), we get:

$$\sum_{s=t}^{\infty} \left( \frac{1}{1+r} \right)^{s-t} (C_s + I_s) = (1+r)B_t + \sum_{s=t}^{\infty} \left( \frac{1}{1+r} \right)^{s-t} (Y_s - G_s)$$

We can re-arrange this equation to read:

$$-(1+r)B_t = \sum_{s=t}^{\infty} \left( \frac{1}{1+r} \right)^{s-t} (Y_s - C_s - I_s - G_s)$$
This says that the present value of future trade surpluses has to be equal to the negative of our current net foreign assets. So if we have foreign debt currently, the PDV of future trade surpluses must be equal to that debt.

Now, suppose that we assume that $\theta = r$. In this case, consumption will be constant over time, so we can just use the budget constraint to solve for first period consumption:

$$C_t = \frac{r}{1 + r} \left[ (1 + r)B_t + \sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} \left( Y_s - G_s - I_s \right) \right]$$

Suppose that $r$ and $\theta$ aren't equal. Then we know that

$$C_{t+1} = C_t \left( \frac{(1+r)}{(1+\theta)} \right)^{1/\sigma}$$

So we can re-write the infinite horizon budget constraint as

$$\sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} C_t \left( \frac{1 + r}{1 + \theta} \right)^{\frac{s-t}{\sigma}} = (1 + r)B_t + \sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} \left( Y_s - I_s - G_s \right)$$

we can re-write the left hand side as

$$C_t \sum_{s=t}^{\infty} \left( \frac{(1 + r)}{1 + \theta} \right)^{\frac{s-t}{\sigma}} = \left( \frac{(1 + r)}{1 + \theta} \right)^{1/\sigma} \sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} \left( Y_s - I_s - G_s \right)$$

applying the formula for an infinite sum on the left hand side
\[
C_t \times \frac{1}{1 - (1 + r)^\frac{1}{1 + \theta}} = (1 + r)B_t + \sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} (Y_s - I_s - G_s)
\]

define \( \gamma \) as

\[
\gamma = 1 - \left( \frac{1 + r}{1 + \theta} \right)^{1/\sigma}
\]

(so \( \gamma \) is negative the growth rate of consumption)

we can re-arrange to get

\[
C_t = \frac{r + \gamma}{1 + r} \left[ (1 + r)B_t + \sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} (Y_s - I_s - G_s) \right]
\]

If \( \gamma = 0 \), we get the same as before. If \( \gamma > 0 \) (that is, consumption growth is negative), then we get consumption at time \( t \) that is higher than what can be sustained permanently. Consumption will fall over time.

Notice that there is a potential problem if the denominator in equation 7 is greater than one - - then the series is not convergent -- this problem arises if parameters are such that people want to have consumption growing faster than the interest rate.....(why, exactly?)

So we will assume that the parameters are such that it is less than one.

(maybe add -- the stuff on annuity values of Y, etc. -- way of showing the PIH).

------------------------

define the permanent level of some variable, \( X \), as , as the constant level that has the same present discounted value as the actual future path. (all this is assuming certainty and constant interest rate).

\[
\sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} X_s = \sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} X_s
\]

so that
for the case where $\Theta = r$, we can re-write the formula for $C$ as simply

$$C_t = rB_t + \bar{Y}_t \cdot \bar{G}_t \cdot \bar{I}_t,$$

substitute this into the current account equation:

$$CA_t = (Y_t - \bar{Y}_t) \cdot (I_t - \bar{I}_t) \cdot (G_t - \bar{G}_t)$$

So this says that the current account just depends on the deviations of $Y$, $G$, and $I$ from their annuity levels. For example, if you have a temporary shock that doesn't affect the annuity value, it shows up purely in the current account. For example, if there is a need for temporarily high investment (say, due to an earthquake), or if there is temporarily high output, etc.

[ don't want to do this, I think... What about the case where $r$ is not equal to $\Theta$? We can look at the expression for consumption in this case, an just re-write it using the notation above for the part that doesn't include the gamma stuff. ]

Now let's look at a more explicit growth model with this same setup (this is appendix A)

$$Y = AK^\alpha$$

$$A_{t+1} = (1+g)^{1-\alpha} A_t$$

So in the steady state of a solow model, output (or output per worker) will grow at rate $g$. (More explicitly -- in the steady state, both output and capital will grow at rate $g$ -- to make this work, it must be the case that technology grows at the rate shown).
Since it is an open economy, the marginal product of capital must always be equal to the world interest rate, r.

\[ r = \alpha A_t K_t^{\alpha - 1} \]

so

\[ K_t = \left( \frac{\alpha A_t}{r} \right)^{\frac{1}{1-\alpha}} \]

We assume that the rate of depreciation is zero. Since capital grows at rate g, it must be the case that investment is just equal to g times the capital stock in period t:

\[ I_t = K_{t+1} - K_t = gx \left( \frac{\alpha A_t}{r} \right)^{\frac{1}{1-\alpha}} \]

Substituting the equation for capital into the production function,

\[ Y_t = A_t^{\frac{1}{1-\alpha}} \left( \frac{\alpha}{r} \right)^{\frac{\alpha}{1-\alpha}} \]

So we can write investment as (using the equations for investment and for output)

\[ I_t = \left( \frac{\alpha g}{r} \right) Y \]

We assume that government spending is some constant fraction, \( \beta \), of output (where \( \beta < 1-(\alpha g/r) \));

So output net of government spending and investment is
Notice that we have this expression at any time \( s \) as just a function of \( Y \) at time \( t \).

We can now find consumption at any time \( t \), using the formula derived above.

\[
C_t = \frac{r + \gamma}{1 + r} \left[ (1 + r)B_t + \sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} (1 + g)^s \left( 1 - \frac{\alpha g}{r} - \beta \right) Y_s \right]
\]

\[
C_t = \frac{r + \gamma}{1 + r} \left[ (1 + r)B_t + \frac{1 + r}{r - g} \left( 1 - \frac{\alpha g}{r} - \beta \right) Y_t \right]
\]

\[
C_t = (r + \gamma)B_t + \frac{r + \gamma}{r - g} \left( 1 - \frac{\alpha g}{r} - \beta \right) Y_t
\]

Substitute this into the definition for the current account, and also substitute the formula above for \( I \) and \( G \),

\[
CA_t = -\gamma B_t \left( \frac{g + \gamma}{r - g} \right) \left( 1 - \frac{\alpha g}{r} - \beta \right) Y_t
\]

Combine the equations

\( B_{s+1} = B_s + CA_s \)

and

\( Y_{s+1} = Y_s (1 + g) \)

with the above equation, and substitute back in the definition of \( \gamma \), and we get:
We will first consider the case where

\[
\left( \frac{1+r}{1+\theta} \right)^{\sigma/\theta} < 1 + g
\]

This says that the desired rate of consumption growth is lower than the growth rate of output.

We can now plot the phase diagram for B, along with the 45 degree line. The intercept is negative, and the slope is less than one. etc.

In the steady state, the consumption/output ratio goes to zero.

if the inequality is reversed, the diagram is reversed. There is still a steady state, but any level above it leads to unlimited accumulation of assets. (what about the case where the two side are equal?).

The steady state B/Y ratio (for either case) is

\[
\left( \frac{B}{Y} \right)^* = \frac{1 - \beta \cdot \alpha g}{r - g}
\]

Notice that it is always negative under the assumptions that we have made.

Notice that at this ratio, the current value of the debt is equal to the entire present value of future Y-I-G !!!

So we can see why it is never possible for the country to have debt that is greater than this limit -- it would be bankrupt! The present value of all of its future cash flow would be smaller than

\[
\begin{equation}
B_{t+1} = \frac{\left( \frac{1+r}{1+\theta} \right)^{\sigma/\theta} B_t}{(1+g)} - \frac{1}{(1+g)(r-g)} \left( 1 - \frac{\alpha g}{r} - \beta \right)
\end{equation}
\]
the value of its debt.

What is the logic here? If desired consumption growth is lower than output growth, then the country will go further and further into debt. In the limit, its consumption will be zero in comparison to its output -- all of output will go to pay the debt from previous consumption. Thus the ratio makes sense.

If consumption growth is higher than output growth -- then it is still possible for there to be a steady state -- but now it is unstable -- it is only a steady state if you start there. Suppose that a country starts with debt equal to the NPV of its entire future cash flow (ie Y - I - G). Then it can just satisfy its FOC for growing consumption and its intertemporal budget constraint in only one way -- to have consumption of zero!

Imagine that it had slightly less debt than this steady state ratio. Then it will have very low consumption, but gradually, the ratio of debt to income would fall (remember that it has preferences to have consumption grow faster than income, so the consumption/income ratio must rise over time). It would move away from the steady state.

(to be added, maybe: continuous time model with adjustment costs to investment?)

Open Economy – How Good an Assumption is it?

So is this how the world works?

Feldstein and Horioka -- test whether Savings and Investment are related. Found very strong relation. This has become known as the "Feldstein Horioka fact." In recent years less true than it was, but still pretty damn strong correlation.

Big industry in explaining the fact without invoking impediments to capital flow.... (Summers?)

There is also an argument that, while we might expect instantaneous flows of physical capital in order to take advantage of differences in marginal products, the same does not hold true for human capital. The reasoning is that, if you lend someone money to buy a machine, if they can't pay you at least you can repossess the machine. In the case of human capital, however, there is no repossession possible. This means that there will not be loans that equalize the return on human capital to the world interest rate. In such a situation, a poor country suddenly opened up to the rest of the world will not instantly have output per capita equal to the world level -- rather there will be a slower adjustment. (see the model by Barro, Mankiw, and Sala-i-Martin, presented above.)

Last important note: One can do lots of other stuff with the theory of the optimal current account,
but it turns out that all of the intuition is exactly the same as the intuition that we will be developing for saving of individuals. What we did for individuals, can be applied (at least in some cases) to countries -- for example the Permanent Income Hypothesis.

END OF PART I
Endogenous Growth Theory

I have introduced the OLG model here because of it is mathematically very similar to the Ramsey and Solow models, and because it is a model that is useful in many contexts. But now I want to forget about the OLG model and ask a more general question about the growth setup that we have been examining: is it right?

A bit of academic sociology: The Ramsey model dates back to the 1920's. The Solow model was done in the 1950's. During the 1960's, growth theory was a big part of work on macroeconomics, and many variations on the basic Solow view were fleshed out (for example, different ways to model technological change). Then, in the 1970's growth theory petered out, and macroeconomics focused more on business cycles. What we have covered so far is all stuff that you might have learned 20 years ago.

Then, starting around 1986, there was a massive increase in the popularity of growth theory -- and this explosion continues today to the point where it sometimes seems like half the papers in macro are on growth theory; whole issues of journals are devoted to it, etc.

People identified with the new theory: Paul Romer, Robert Lucas

Reasons for the explosion

1. Theoretical problem with the Solow model (exogenous technological change; Solow residuals)

2. Availability of new data -- Summers and Heston -- that allowed testing of Solow model.

3. Exhaustion of business cycle research (agreement on many points) combined with long period w/out recessions.

4. Intellectual changes: Development of new tools by economists that allowed modelling of endog growth stuff.

5. Realization of relative importance of growth as compared to cycles -- driven home by the productivity slowdown.
Let me first amplify some of these problems

-----------------------------------------------
Solow Residual

One of the potential theoretical problems with the Solow model is that technological growth is taken to be exogenous. To decide whether this is a problem, we might want to know how big technological growth is. We have said that in the steady state of the Solow model, total output grows at the rate \( n + g \) -- and since we know the rate of growth of total output (that is, \( Y \)), and we know \( n \), we can back out \( g \). But what if we are not in the steady state?

Robert Solow provided the answer to this problem. It is important to note that the technique that he provides is not dependent on the other parts of the Solow model (for example the exogenous saving rate).

Start with the production function:

\[
Y = A \cdot F(K, L)
\]

Where \( A \) is a measure of the state of technology and \( F() \) is some CRS function that combines capital and labor. \( A \) is called "total factor productivity."

[Note that another commonly used measure of productivity is \( Y/L \) -- that is, output per worker. While this does get a some measure of economic well being, it does not pretend to measure technology. One can easily increase \( Y/L \) by simply raising the amount of capital in the country, without necessarily inventing any new methods of production. By contrast, total factor productivity cannot be changed just by changing the amount of inputs.]

We can totally differentiate to get

We differnetiate w.r.t. time to get

\[
dY/dt = dA/dt \cdot F(K, L) + A \cdot F_L(K, L) \cdot dL/dt + A \cdot F_K(K, L) \cdot dK/dt
\]

now divide the left and right sides by the corresponding sides of the production function to get:

\[
\frac{dY/dt}{Y} = \frac{dA/dt}{A} + \frac{AF_L(K, L)dL/dt}{AF(K, L)} + \frac{AF_K(K, L)dK/dt}{AF(K, L)}
\]
We now multiply top and bottom of the second term on the left by L, and do the same with K in the third term. We use a hat over a variable to indicate its growth rate, that is \( \frac{dx}{dt}/x \)

\[
\hat{Y} = \hat{A} + (1 - \alpha)\hat{L} + \alpha\hat{K}
\]

where \( \alpha \) is just capital's share of output.

This equation shows that there are three reasons that output can grow: increase in capital, increase in labor, and improvement in technology.

Using this equation, we can easily figure out the rate of growth of technology, which is just \( \hat{A} \) hat. All of the other pieces of the equation (growth rates of capital, labor, and output, and capital's share of output) are, in principle, observable.

Conducting this exercise for the US in the postwar period gives values like: (all in % per year) (assuming \( \alpha = .3 \))

<table>
<thead>
<tr>
<th></th>
<th>( \hat{Y} )</th>
<th>( \hat{L} )</th>
<th>( \hat{K} )</th>
<th>( \hat{A} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1948-60</td>
<td>3.5</td>
<td>1.1</td>
<td>3.5</td>
<td>1.6</td>
</tr>
<tr>
<td>1960-73</td>
<td>4.7</td>
<td>1.8</td>
<td>4.6</td>
<td>2.1</td>
</tr>
<tr>
<td>1973-90</td>
<td>2.7</td>
<td>2.0</td>
<td>3.2</td>
<td>0.1</td>
</tr>
</tbody>
</table>

(source: CBO, oct 1994)

So we can see that in the 70's and 80's, something bad has happened to the rate of technological progress. In fact, all of the slow growth in these two decades seems to be due to this factor.

Further, the productivity slowdown has not been limited to the US. All of the industrialized countries seem to be having one.

This slowdown in the growth of technology is potentially the most serious economic problem that we face...

[to be added: conceptual problem with growth accounting. Think about the steady state in the Solow model. Accounting procedure will attribute only some of growth to technology, while in a logical sense all of it is due to technology.]
Lucas’ calculation of the relative importance of long- and short-run. (Models of Business Cycles, Yrjo Jahnsson Lectures, published 1987). Start with the usual discounted utility function

\[ U = E \left[ \sum_{t=0}^{\infty} \beta^t \frac{c^{t-\sigma}}{1-\sigma} \right] \]

and a stochastic process for consumption that has both a deterministic trend and variation around the trend:

\[ c_t = (1 + \gamma)(1 + \mu) \left( \frac{z_t}{e^{\frac{1}{2}\sigma^2}} \right) \]

\[ \ln(z_t) \sim N(0, \sigma^2_z) \]

The term in parentheses is a lognormally distributed multiplicative error with a mean of one.

Assigning appropriate values for the trend growth rate of consumption and the variance of consumption (both averages of the post WWII US economy), and choosing a value for \( \beta \) and \( \sigma \), one can calculate the expected value of consumption. Lucas then asks two questions. First, what is the variation in \( \gamma \) that would be equivalent to eliminating consumption variability entirely? Answer: .00042 (.04 percent). Second, what would be the variation in \( \gamma \) that would be equivalent to raising the growth rate (\( \mu \)) by one percent? Answer: .17. Conclude: cycles are unimportant.

What might be wrong with this sort of calculation? One problem is that it ignores heterogeneity – in a recession it is not that each person has a 5% consumption deline, but that 10% of people have a 50% decline. But note that heterogeneity alone is not enough to convince
us that cycles are bad – after all, in a boom the unemployment rate is low....

Note: habit formation is almost certainly part of the explanation for why people care so much about cycles relative to growth.

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2. A second problem with the Solow model comes from its predictions about differences in the marginal product of capital across countries. (see homework problem)

Assuming that everyone has the same technology and that capital's share is 1/3, a difference of a factor of 5 between the level of output per capita between two countries should mean that their marginal products of capital differ by a factor of 25. If the interest rate is

\[ r = f'(k) - \delta \]

Then, say, for the US the real rate were 5% and depreciation were 5%, then \( f'(k) \) in the US would be 10%, and in a poor country it would be 250% (and the interest rate would be 245%)!!! This seems pretty much larger than real interest rates that we observe in other countries. Nor do we observe the flows of capital that would be expected in response to such large differences in interest rates. [All of this is in the King and Rebelo model on the reading list].

Note that you can get around this problem by assuming that different countries have different technologies. But you have to believe that technologies are really different to produce the differences in output observed. [homework question: can calculate how many years behind other country must be]

3. A third problem with the Solow/Ramsey model that emerged when better data on cross country growth became available was the “convergence” problem: Assume that every country has the same utility function, discount rate, depreciation rate, and production technology. Suppose that we then look at all of these countries, and notice that they differ in income per capita. How does the Solow model explain such differences? Clearly, it must be that they differed in their initial level of capital. For example, if you looked at Germany and Japan after the war, they had very low levels of capital, and so low output.

What does the Solow/Ramsey model say should happen in such a case? It says that the poor countries should grow faster than the rich countries (remember that all countries grow because of technological progress). So if we graphed the growth rate of income per capita against the initial level of income per capita, we should see a downward sloping line.

In fact, we see nothing of the sort-- rather we see a big mess with no particular slope. This is the failure of convergence.

So there are three strikes against the Solow model
- exogenous tech change
- interest differentials
- convergence
Endogenous growth models

[Note for future drafts: could make the production function in all of the this or earlier material $y=Ax^a$ instead of $y=k^a$ -- the $A$ is just a constant that does not affect any of the qualitative results. Doing it this way has the advantage of making it consistent with the endogenous growth section]

We now take up endogenous growth models that solve some of the problems described above. We start with the simplest of the endogenous growth models, then move on to more complicated ones.

Start with the Solow growth model, using a Cobb Douglas production function and no technological change. $Y=Ax^a$, where $A$ is a constant measuring the state of technology. The usual differential equation is:

$$\dot{k} = sAk^\alpha - (n+\delta)k$$

we can re-write this as:

$$\frac{\dot{k}}{k} = sA(k^{-1})k^\alpha - (n+\delta)$$

As before We can graph the two parts of the right hand side of this equation.

[picture]

The difference between these two is $\dot{k}/k$, the growth rate of the capital stock. What about the growth rate of output? To see the relation between these, takes logs and differentiate the production function:

$$\frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k}$$

So the growth rate of output is just proportional to the growth rate of capital.

So what happens to the growth rate of output per capita away from the steady state? If we are below the steady state, the growth rate of output monotonically decreases as we approach the
steady state.

What happens to the growth of output if we increase saving. in this case, the $sA k^{\alpha-1}$ curve shifts up, we have a temporary increase in growth, then growth peters out to zero again.

Now consider what happens to this picture as we raise $\alpha$. You should see that the downward sloping line gets less sloped.

To see the effect of this, imagine starting in a steady state, then raising the saving rate by the same amount in two economies which have different values for $\alpha$. Initially, the two economies have similar growth rates of capital. [you can check this by looking back at the formulas for convergence speed. The bigger is $\alpha$, the longer is the half life of differences. Specifically, the half life of the difference is $0.7 / [0.5 - (n + g)]$ -- but in the case where $\alpha$ is low, growth peters out quickly, while in the case where $\alpha$ is high, growth lasts a long time, and it takes longer to get to the new steady state.

Now think about the limiting case, where $\alpha=1$. In this case, growth never stops.

$$\frac{\dot{y}}{y} = \frac{\dot{k}}{k} = sA -(n + \delta)$$

So here growth is endogenous, in the sense that it is determined by the other parameters in the model (not exogenously).

This is the simplest endogenous growth model, sometimes called the Ak model, since the production function is

$$y = Ak$$  (ie cobb douglas with $\alpha=1$)

Several things to notice about the model:

1). What does it say about interest rate differentials? Well, $f'(k)$ is just $A$, and so $r=A-\delta$. So what should the interest rate differential be between rich and poor countries? Nothing. This is a big improvement over the huge differentials implied by the Solow model.

2.) Suppose that we looked at data on the determinants of growth rates of countries. What does this model say about the relation between a country's level of income and its growth rate? It says that there is no relation! Growth is just a function of the level of saving and of $(n+\delta)$. This seems to be a good property, because as we saw earlier, there is no relation visible when you just plot growth and initial income.
3) What about saving? The model says that higher saving will be associated with higher growth.

4) n? Model says higher n leads to lower growth. Note in the regular Solow model, n and s affect level but not steady state growth rate.

In a moment we will talk about this production function, but first lets do the Ramsey version of the Ak model. The Hamiltonian for the social planner is:

$$H = e^{\theta t} \frac{c^{1-\sigma}}{1-\sigma} + \lambda (Ak - c - (n + \delta)k)$$

Solving through all of the usual steps give the equation of motion for consumption

$$\frac{\dot{c}}{c} = \frac{A - \delta - n - \theta}{\sigma}$$

(Alternatively we can just get this from the same $\dot{c}/c = (1/\sigma)(r - n - \theta)$ logic as before).

Note a groovy thing about this equation: it doesn't depend on the level of capital in the economy!

What about the growth rates of k and y?

We know that $\dot{y}/y = \dot{k}/k$

And we can re-write the differential equation for capital to be

$$\frac{\dot{k}}{k} = A - \frac{c}{k} - (n + \delta)$$

so substitute $\dot{y}/y$ for $\dot{k}/k$, and replace the k with y/A

$$\frac{\dot{y}}{y} = A - \frac{c}{y} - (n + \delta)$$

Now look at this equation for a moment. If $\dot{y}/y = \dot{c}/c$, then (c/y) will be constant, and so $\dot{y}/y$ will be constant. If $\dot{y}/y > \dot{c}/c$, then c/y is falling and $\dot{y}/y$ is rising. In this case, $\dot{y}/y$ will
remain larger than $\dot{c}/c$, and $c/y$ will fall toward zero. This is clearly not the optimal path. Similarly, if $\dot{y}/y < \dot{c}/c$, then $c/y$ will rise continually, eventually going over 1, which is not feasible. So, $\frac{\dot{c}}{c} = \frac{\dot{y}}{y} \left( \frac{k}{k} \right)$.

And all of these growth rates are determined by the preference parameters.

Note that all of this is true all of the time -- not just in a steady state. Put another way, all of the time is a steady state. There are no "transitional dynamics," as there are in the Ramsey model. Given some level of output, you just start growing at a rate determined by your parameters, and keep at this growth rate forever.

So two countries with the same parameters but different starting points would remain the same percentage difference apart forever.

[picture]

And two countries with different parameters (say different discount rates) would grow at different rates forever -- and would end up arbitrarily far apart in the long run. So if Japan is growing faster than the US now, there is no reason why it shouldn't continue to do so once it passes the US.

You can also solve for the saving rate explicitly given the $\dot{c}/c$ and $\dot{y}/y$ equations:

$$\frac{\dot{c}}{c} = \frac{A \cdot (n + \delta) - \theta}{\sigma} = \frac{\dot{y}}{y} = A - A(\frac{c}{y})(n + \delta) = A - A(1 - s)(n + \delta) = sA -(n + \delta)$$

Notice that there is the same relationship between growth and saving that we saw before.

Continuing to solve gives:

$$s = \frac{1}{A \sigma} \left[ A + (\sigma - 1)(n + \delta) - \theta \right] = \frac{1}{\sigma} \left[ 1 + \frac{(\sigma - 1)(n + \delta)}{A} - \theta \right]$$

It is clear that $ds/d\theta$ is negative: if you care about the future less, you will save less.
The derivative of saving with respect to $\sigma$ (skipping a few intermediate steps) is

$$\frac{ds}{d\sigma} = -\sigma^2 \left[ \frac{1 - n + n + \delta + \theta}{A} \right]$$

This is negative iff $A > (n + \delta + \Theta)$, which is the condition for consumption (and output) growth to be positive. The intuition is that, if consumption is being chosen such that it is growing over time, then an increase in risk aversion makes the marginal utility of consumption lower in the future, and so makes consumption growth less desirable. So the saving rate will fall. If consumption was initially falling, then an increase in the coefficient of relative risk aversion will make the marginal utility of consumption higher in the future, and so will raise the saving rate. [remember in the original Ramsey model, high $\sigma$ meant that the transition from an initial capital stock to the steady state was slow -- this is the same intuition, but it lasts forever].

Finally, take the derivative of the saving rate with respect to $A$:

$$\frac{ds}{dA} = -\frac{1}{\sigma^2} \left[ \frac{(\sigma - 1)(n + \delta) - \theta}{A^2} \right]$$

This can be either positive or negative for reasonable sets of parameters. For example, if $n + \delta = \Theta$, then it will be zero for $\sigma = 2$ and negative for higher values of $\sigma$. Indeed, the higher is $\sigma$, the more likely it is to be negative: the reason is that for big $\sigma$ the consumption smoothing effect dominates. Note that having $ds/dA < 0$ seems to contradict the spirit of endogenous growth models (see Carroll, Overland, and Weil for more on this point).

So in general the Ramsey version of this model is just like the Solow version (without the transitional dynamics that the usual Ramsey model has) -- it's only advantage is that it tells us the saving rate explicitly as a function of the parameters: Raising $\Theta$ or $\sigma$ lowers the saving rate.

so now we have the question: why should we believe in this production function anyway?

How can we reconcile this type of model where $\alpha = 1$ with the observation that capital's share of output is just $1/3$?

Two stories:
First story for endogenous growth: externalities.

remember externalities from micro: If you have the apple orchard next to the place where they keep the bees, etc.

Take the production fn \( Y = A K^\alpha L^\beta \)

Now lets imagine that \( A \), the state of technology in the economy, is a function of the amount of capital.

Why this might be: Well you might think that the more investing that has been done in the past, the more we would have learned about how to do it right. This is story about "learning by doing" which traces back to Kenneth Arrow. Well the total amount of investment that has been done in the past is -- assuming no depreciation -- is just \( K \).

So if we made \( A \) a function of capital per person: \( A = B^*(K/L)^\gamma \), then the production function would be

\[
Y = B^* K^\alpha L^\beta (k)^\gamma \quad \text{(where } k = K/L)\]

If we assume that \( \alpha + \beta = 1 \), we get constant returns to scale for firms. (note that for a firm, the economy wide level of \( k \) is taken as given). This is good for two reasons: first, it seems to accord with what we see in the world: that big firms are not much more efficient than small ones. Second, as you will see in micro, if firms do not have CRS production functions then perfectly competitive general equilibrium does not exist. This latter problem -- which is a limitation of theory -- can be gotten around in various ways, but we don't deal with it here.

[A side note, if we had written the externality as a function of total capital, rather than capital per worker, we would have gotten the aggregate production function:

\[
Y = B K^\alpha L^\beta + \gamma L^\beta
\]

In this case, there are IRS at the national level -- that is doubling the size of the country more than doubles output, which seems like a silly story.]

We can write the production function in per capita terms as

\[
y = B^* k^\alpha k^\gamma
\]

So clearly, if \( \gamma + \alpha = 1 \), then we have the Ak model again, with \( A = B \).

[I now set \( B = 1 \), because it is not interesting]
Now one of the things that you learn in micro is that in the presence of externalities, the competitive equilibrium is not efficient. The reason is that people and firms do not take externalities into account in making their decisions.

In this case, the firm takes the economy-wide level of $k^γ$ as fixed, and so it's production function is

$$y_i = k^γ k_i^α$$

or in non-per capita terms: $$Y_i = k^γ K_i^α L_i^{1-α}$$

Notice that from the firm's perspective, labor is productive, and so labor earns a share $(1-α)$ of total output.

The marginal product of capital from the perspective of a firm is

$$\text{MPK (firm)} = α k_i^{α-1} k^γ$$

assuming that the economy is made up of identical firms (and so they all have the same capital/labor ratios)

$$\text{MPK (firm)} = α k^{α+γ-1}$$

While the social planner takes into account the externality of capital, and so sees the marginal product as

$$\text{MPK} = (α+γ) k^{α+γ-1}$$

So in the competitive equilibrium there will be too little investment in capital.

So this story produces an endogenous growth model if $α+γ=1$. But note that given capital's share of income is 1/3, this means that there are really large externalities that must be present.

**Second story for endogenous growth model: Human Capital**

We discussed human capital earlier. Now we look at an endogenous growth model based on it:

**Lucas Growth Model**
The Lucas article on the reading list goes through a model of growth with human capital.

Production fn: \( Y = K^\alpha (u^\alpha h^\alpha L)^{1-\alpha} \)

Where \( L \) is labor, \( h \) is the amount of human capital per person, and \( u \) is the fraction of their time that people spend working.

or in per capita terms

\[ y = k^\alpha (uh)^{(1-\alpha)} \]

So production is CRS in capital and this quality and time adjusted labor input.

capital has the usual differential equation: \( \dot{k} = y - c - (n + \delta)k \)

But now there is also a production function for human capital. To produce human capital, you have to take time away from producing goods. So \((1-u)\) is the fraction of work time spent producing human capital,

\[ \dot{h} = \phi h(1-u) \quad \text{or alternatively} \quad \frac{\dot{h}}{h} = \phi(1-u) \]

This says that human capital is produced by a constant returns to scale production function, where \( \phi \) is just some constant. That is, if we double the amount of input into producing human capital \((h^\alpha(1-u))\), then we double the amount of human capital produced.

[notice that we treat human capital as not being depreciating, and as not being diluted by the arrival of new people. Thus in this formulation, human capital is really like knowledge; and creation of new human capital is really like research and development. Allowing for a \(-(n+\delta)h\) term in the equation would not really change things that much.]

Lucas goes on to solve this as a social planner’s problem using a Hamiltonian. There are two state variables \((h \text{ and } k)\) and two control variables \((u \text{ and } c)\).

For simplicity, we will do the “Solow” version of the model, in which \(u\) and \((s=(y-c)/y)\) are taken to be exogenous. Recall that the steady state of the Ramsey model looks just like the steady state of the Solow model in terms of output and saving being constant, and output being a function of the saving rate (the only difference between the two models being that for the Ramsey, we can find the saving rate as an explicit function of the taste parameters). Here, the analogue of the steady state is the “balanced growth path” (see below). The balanced growth path of the Lucas model looks just like the balanced growth path of the model presented here: \(u\) and \(s\) are constant, and the growth rate of output is a function of them. The only differences
between the true Lucas model and the one presented here are in the dynamics off of the balanced growth path, and that in the Lucas model \( u \) and \( s \) are explicit functions of the taste parameters.

Re-write capital accumulation using \( s \) instead of \( c \):

\[
\dot{k} = sy - (n+\delta)k
\]

so

\[
\frac{\dot{k}}{k} = s\frac{y}{k} - (n+\delta) = sk^{\alpha-1}(uh)^{\alpha} - (n+\delta) = s\left(\frac{k}{h}\right)^{\alpha-1}u^{\alpha} - (n+\delta)
\]

Now we draw a diagram with both \( \dot{k} \) and \( \dot{h} \) on the vertical axis and \( k/h \) on the horizontal axis. The \( \dot{h} \) equation is just a horizontal line. The \( \dot{k}/k \) equation slopes downward.

What happens to \( k/h \) over time? Where the two curves cross, \( k \) and \( h \) are growing at the same rate, and so \( k/h \) will remain constant. Call this \( (k/h)^* \), the balanced growth ratio of \( k \) to \( h \). For lower levels of \( k/h \), \( k \) will be growing faster than \( h \), so \( k/h \) will be growing over time. For \( k/h \) bigger, vice versa. So whatever level of \( k/h \) you start with, eventually it will move to \( (k/h)^* \).

What about the growth rate of output? Starting with the production function, we can take logs and differentiate:

\[
\ln(y) = \alpha \ln(k) + (1-\alpha) \ln(u) + (1-\alpha) \ln(h)
\]

\[
\frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k} + (1-\alpha) \frac{\dot{h}}{h}
\]

So in the long run (along the balanced growth path), the growth rates of output, capital, and human capital are all equal, and are all determined by the \( \dot{h}/h \) equation. The lower is \( u \), the faster is growth along the balanced growth path.

Unlike the Ak model, there are transitional dynamics, which depend on the initial \( k/h \) ratio: if \( k/h \) is initially lower than the balanced growth level, then output will initially grow faster than it will in the long run.

Do the experiment of lowering \( u \) from the steady state. What do time series look like? The level of \( y \) initially falls, but the growth rate jumps, then continues to rise.
Summary of endogenous growth: there are many ways to go, the key is that you have constant returns to the accumulable (that is, the non-labor) factors. In the externalities model, the factor is capital. In the Lucas model, h and k are the factors that can be accumulated, and there are constant returns to the two of them, that is, if you double h and double k, you double output. By contrast, in the regular model, there is decreasing returns to capital.

So now we know how to write down models in which growth is endogenous. Question: are they right?

Last time I said that the three strikes against the solow model

-exogenous tech change
-interest rate differentials
-failure of convergence

The first problem is a theoretical one, and I am not going to be able to fix it. Romer book has good discussion of endogenous technological change. The problem is that since all of the countries in the world have the same technology, you can't really test models of technological change (that is, they don't have any implications for differences between countries).

But we can look a little more closely at the other two.

-- interest rate differentials

Here, we can see that the problem is not too hard to fix if we take a page from the endogenous growth book and consider human capital. Mankiw, Romer, and Weil examine data from a cross section of countries.

Recall the version of the Solow model with both human and physical capital that we examined earlier.

\[ Y = A K^\alpha H^\beta L^{1-\alpha-\beta} \]

We solved for the steady state level of output per efficiency unit:

\[ y = \frac{Y}{eL} = (n + g + \delta) \frac{\alpha^+\beta}{1-\alpha-\beta} \frac{\alpha}{\tau^\alpha} \frac{\beta}{\delta^\beta} \frac{1}{\delta^\beta} \frac{1}{\alpha^+\beta} \]
We can put this back in non-efficiency unit terms. Then taking logs:

\[
\ln \left( \frac{Y}{L} \right) = \ln(A(0)) + gt - \frac{\alpha + \beta}{1 - \alpha - \beta} \ln(n + g + \delta) + \frac{\alpha}{1 - \alpha - \beta} \ln(s_t) + \frac{\beta}{1 - \alpha - \beta} \ln(s_h)
\]

One can run this regression, and back out values for \( \alpha \) and \( \beta \) (note that MRW also run other regressions that produce values for \( \alpha \) and \( \beta \) - these will be discussed later).

MRW conclude that, taking human capital into account, a good approximation to the production function is

\[
Y = A K^{\alpha} H^{\beta} L^{1-\alpha-\beta}
\]

where \( \alpha = \beta = 1/3 \)

In this case, \( H \) is supposed to mean schooling (which is how we measure it), not knowledge.

In this case, half of what we were thinking of as labor's share is really human capital's share. So if we lump all capital together (that is, if the production functions for human and physical capital are the same), we can just think of capital's share being 2/3. In this case, as shown on the homework, the interest rate differential problem goes away. Countries that differ in income by a factor of 5 differ in MPK by a factor of only \( \sqrt{5} \).

As for the third strike -- the convergence regressions -- let's examine these a little more closely.

What does the Solow model really say about convergence.

There are two reasons countries can have different output -- different initial conditions or different values of \( n \) and \( s \). When we said that we expected convergence, that is only the case when countries have the same steady state and different initial conditions. If the reverse were true -- if they had different steady states, but were all in them already, then everyone would have the same growth rate no matter what their level of income per capita.

So growth is higher when \( y \) is below \( y^* \)
\[
\dot{y}/y = f(y^* - y)
\]

So what happens if you consider a regression of \( \dot{y}/y \) on things like the initial level of income, saving rate, and population growth?

\[
\dot{y}/y = \beta_0 + \beta_1 s + \beta_2 n + \beta_3 y
\]

\( s \) and \( n \) determine the steady state, and the higher is the steady state, holding the current level of income constant, the higher is growth. So \( \beta_1 \) is positive and \( \beta_2 \) is negative.

Says that holding \( s \) and \( n \) constant (that is, holding \( y^* \) constant), higher \( y \) leads to lower growth, so \( \beta_3 \) is negative.

The result on \( \beta_3 \) is called conditional convergence -- controlling for everything else, coefficient on income is negative.

What about endogenous growth models? In general these say the same thing for \( \beta_1 \) and \( \beta_2 \), but they say that \( \beta_3 \) is zero.

Go to the data -- solow wins big.

So conditional converge is not only not a strike against Solow, it is a strike against endogenous growth.

[add: What is the interpretation of the "convergence" regression when we add other stuff to the RHS -- black market exchange premium, etc. Are these things good for growth, or good for level].

---

**Knowledge spillovers in the Lucas setup.**

Consider the following two-country version of the Lucas growth model.

\( y_i = \) output per capita in country \( i \) \( (i = 1,2) \)

\( k_i = \) capital per capita in country \( i \)

\( n = \) population growth \( (\)assumed to be the same in both countries\() \)

\( \delta = \) depreciation \( (\)assumed to be the same in both countries\() \)

\( s = \) saving rate \( (\)same in both countries\() \)

\( u_i = \) fraction of their time that people spend working in country \( i \)

\( (1-u_i) = \)fraction of their time that people spend building human capital in country \( i \)
Production:
\[ y_i = k_i^\alpha (u_i h_i)^{(1-\alpha)} \]
capital accumulation:
\[ k_i = s y_i - (n+\delta)k_i \]

So far all of this is like the model presented in class. For human capital accumulation, we add a new assumption. Let \( h_l \) be the level of human capital per capita in the country that has a higher level of human capital (the "leader"), and \( h_f \) be the level in the country that has less human capital (the "follower.")

We assume that human capital production in the leader country works exactly as in Lucas' model:
\[ \dot{h}_l = \phi (1-u_l)h_l \]

Human capital is produced in the follower country by two methods: first, there is the same production as in the leader country. But second, human capital (which in this model is the same as "knowledge") spills over from the leader country to the follower country. The amount of this transfer depends on the difference in their levels of knowledge.
\[ \dot{h}_f = \phi (1-u_f)h_f + \beta (h_l - h_f) \quad \beta > 0 \]

Assume that the two countries have the same saving rate, but \( u_1 < u_2 \). In other words, in country 2 people spend a larger fraction of their time working (and a smaller fraction producing human capital) than in country 1.

A) Describe the steady state of the model. Solve for each country's growth rate in steady state.

Obviously country 2 will be the technology follower. So we can solve for country 1 exactly as if it were alone in the world. Now the key question is what do the country's growth rates of \( h \) look like? Answer: they must be the same: if not, then either country 2 would catch up or fall infinitely far behind.
(Figure: put \( h_2/h_1 \) on the horizontal axis, and the two \( h \) dot lines on the vertical)

B) Solve for the relative level of human capital per person in the two countries in steady state. How do the parameters \( \phi \) and \( \beta \) affect the ratio of \( h_1/h_2 \).
Set the two $\dot{h}/h$ equations equal. Then ..... 

C) (hard) Solve for the relative level of consumption in the two countries in steady state. How (and why) do the parameters $\phi$ and $\beta$ affect the relative level of consumption.

Extending this idea:

Basu, Feyrer, and Weil. Instead of doing the “Solow” version of this model, we can do the “Ramsey” version, in which we take $u$ as a decision variable. (For simplicity we don't think about changing $s$; in fact, in the paper, we take $s$ as given.)

First, think about the one-country case. Suppose that there is a social planner who cares about discounted consumption. Then clearly if he has a higher value of $\Theta$, there will be less R&D, slower growth, but higher initial consumption.

Now, consider the case of a world with two countries. Suppose, for simplicity, that R&D in one country is fixed. Suppose that I am a social planner, and I was considering doing exactly the same amount of R&D as the other guy is already doing. My optimal path will now be to do _less_, and let him be the leader.

It is even possible that I might have wanted to grow faster than him, but if his is fixed, I will let him be the leader.
(All of this gets into dynamic games, which are too hard for me....)

Models of Technological Progress

[Note: we do not cover these in 2005-06, since Peter Howitt himself is teaching in the second semester]

Aghion and Howitt

(This is the “basic model” from their book)
Production Setup

Production function for final output

\[ y = Ax^\alpha \]

where \( x \) is an intermediate good. Output is produced by perfectly competitive firms, that take \( x \) as their only input. Good \( x \) is in turn produced by a monopolist, who uses labor as his only input. The production function for \( x \) is that one unit of labor produces one unit of the intermediate good. This production function for the intermediate good never changes.

Good \( x \) is specific to a technology that is represented by \( A \). The monopolist holds the patent and faces no competition.

The wage at time \( t \) is \( w_t \)

We can solve for the monopolist’s optimal price, production, and output.

Since the final goods sector is competitive and takes only \( x \) as an input, the price of \( x \), \( p_x \), is just its marginal product

\[ p_x = \alpha A x^{\alpha-1} \]

The monopolist takes this as the demand curve that he faces. His marginal cost is \( w_t \). So his profit, \( \pi \), is

\[ \pi = p_x x - w x \]

We can solve this by substituting in the demand curve, and we get the optimal price

\[ p_x = \frac{w}{\alpha} \]

We can also solve for optimal quantity of \( x \)

\[ x = \left( \frac{\alpha A}{w} \right)^{\frac{1}{1-\alpha}} \]

Production of \( x \) is the same as the labor used in producing \( x \). Defining \( \omega \) (small omega)
as the ratio w/A, we have

\[ x = x(\omega) \]

which is a decreasing function.

Finally, profit is

\[ \pi_t = A \left( \frac{1}{\alpha} - 1 \right) \alpha^{\frac{2}{\alpha}} \left( \frac{w}{A} \right)^{\alpha^{\frac{1}{\alpha}}} \]

defining \( \omega \) (small omega) as the ratio w/A, we have

\[ \pi_t = A_t \pi(\omega) \]

where \( \pi \) is a decreasing function (in the Aghion and Howitt book this function has a tilde on top of it, but I can't get my word processor to do that.)

**Innovations**

We index innovations by \( t \) – so note that \( t \) is not time. You can think of \( t \) as indexing the generation of technology (or the "epoch") that is currently in place.

\( A_t \) is the technology after innovation \( t \). New innovations raise the level of \( A \) by a factor \( \gamma \):

\[ A_{t+1} = \gamma A_t \]

Thus the form of technological progress in this economy is new generations of this intermediate goods and associated new values of \( A \).

An innovation gives a monopoly right to the owner of the innovation until the next innovation.

**Labor Market**
Workers can produce the intermediate good or do R&D (not that no labor is used in production of the final good.) Innovations are produced by labor alone. The labor force is fixed at L. Let n be the number of workers doing R&D. Thus:

\[ L = n + x \]

**Technology Production**

The arrival rate for innovations is \( n \lambda \) – that is, this is the probability that a new innovation will arrive.

where \( n \) is the number of workers doing R&D, and \( \lambda \) is the probability of a worker coming up with an innovation. (Note that there is no possibility of simultaneous innovations, so doubling the workers doubles the rate of arrival of new innovations.)

Firms that are doing R&D use only labor. They are also risk neutral, so their spending on research has the simple arbitrage condition

\[ w_t = \lambda_t V_{t+1} \]

We take the interest rate \( r \) as exogenous and constant. What is the value of holding an innovation? Call \( \pi_{t+1} \) the profits that are earned by a firm that holds patent \( t+1 \). These are enjoyed for as long as you hold the monopoly. Since \( \lambda_{t+1} n_{t+1} \) is the rate at which new innovations will arrive, this gets factored into the value

\[ V_{t+1} = \frac{\pi_{t+1}}{r + \lambda_{t+1} n_{t+1}} \]

(Could derive this starting with the PDV integral)

**Solution**

We can take the arbitrage condition, substitute in for \( V_{t+1} \) and for \( \pi_{t+1} \) to get

\[ w_t = \lambda A_{t+1} \frac{\pi(\omega_{t+1})}{r + \lambda_{t+1} n_{t+1}} \]

Dividing both sides of this equation by \( A_t \) gives
\[ \omega_t = \lambda \frac{\gamma \pi(\omega_{t+1})}{r + \lambda n_{t+1}} \]  \hspace{1cm} (1)

we can also rearrange the labor market equation to be

\[ x(\omega_t) = L = n_t \] \hspace{1cm} (2)

Notice that in the first of these equations, \( \omega_t \) depends on future values of \( \omega \) and \( n \) – the reason is that the value of doing R&D today depends on how long you expect to hold onto your monopoly in the future.

We examine only steady states, where \( \omega \) and \( n \) are constant (Aghion and Howitt consider out of steady state stuff, I think.) So equations 1 and 2 become two equations in two unknowns. We can graph them, with \( n \) on the horizontal axis and \( \omega \) on the vertical. There is always an intersection at some value of \( n \) less than \( L \). This is the equilibrium.

Notice that in equilibrium \( n \) will be constant, but growth will still be stochastic, since the time to a new innovation is stochastic.

Comparative statics:
- increase \( L \): this will shift the curve for (2) outward: higher value of \( n \)
- lower \( R \): This will shift the curve for (1) upward: higher value for \( n \) (and \( \omega \))

---

**Technological Progress Through the Creation of New Intermediate Goods**

Here is a second model of technological progress: in this case, progress takes place through the creation of new intermediate goods. Model is that of Grossman and Helpman. This presentation based very heavily on Barro and Sala-i-Martin.

Setup of the model:

- no capital
- constant quantity of labor: L
- \( Y \) is output, produced with labor and intermediate goods, X
- \( N \) is the number of intermediate goods.

Intermediate goods are not capital -- they are measured as flows (produced using output, used in producing output).
i indexes firms
j indexes intermediate goods (j=1...N)

production function for a firm producing final output:

\[ Y_i = L_i^{\alpha} \left( \sum_{j=1}^{N} X_{i,j}^{\alpha} \right)^{\alpha} = L_i^{\alpha} X_{i,1}^{\alpha} + L_i^{\alpha} X_{i,2} + \ldots \]

note:
-- use of each intermediate good does not affect the productivity of the others.
-- there is decreasing marginal productivity of each intermediate good [picture]
-- So: the firm will use all intermediate goods in the same quantity.

Call \( X_i \) the quantity of each intermediate good used by firm \( i \)

\[ Y_i = L_i^{\alpha} N X_i^{\alpha} = L_i^{\alpha} (N X_i) \alpha N^{\alpha-1} \]

note that \( N X_i \) is the total input of intermediate goods.

so there is CRS to inputs of L and intermediate goods (ie L and \( N X_i \)). But raising \( N \) (holding \( N X_i \) constant -- that is, not changing the total quantity of intermediate goods) raises output. The reason can be seen by thinking about the productivity of a given intermediate good in the firm's production function: Bigger \( N \) allows the firm to use less of each one, and this raises the average product for the part of production that used that intermediate good.

We assume that the firms that produce final output (\( Y \)) using labor and intermediate goods are perfect competitors -- they have CRS and will earn zero profits (that is, euler's theorem will hold).

What are their demands for inputs?

\[ \text{profit}_i = Y_i - wL_i - \sum_{j=1}^{N} P_j X_{i,j} \]
so the firm's demand is

\[ X_{i,j} = L_i \left( \frac{\alpha}{P_j} \right)^\frac{1}{1-\alpha} \]

and economy-wide demand is

\[ X_j = \sum_i X_{i,j} = \left( \frac{\alpha}{P_j} \right)^\frac{1}{1-\alpha} \sum_i L_i = L \left( \frac{\alpha}{P_j} \right)^\frac{1}{1-\alpha} \]

Technological change in this model takes place via the expansion in the number of intermediate goods.

This is supposed to be a metaphor for technological progress more generally -- in real life some technological progress takes place via the creation of new final goods, and some takes place through the replacement of old intermediate goods with new ones, and some takes place through other improvements in the production process. The point is that we want a way to model technological progress where inventors earn back the cost of doing inventing, and this model of intermediate goods supplies such a model.

So: inventors create new intermediate goods in order to reap future profits -- otherwise they wouldn't do it. Note the special nature of creating new technologies (as opposed to creating goods or services): technologies are "non-rival." If I figure out a new way to do some task, your using it does not diminish my ability to use it. The non-rivalry of technology means that there is a big social benefit to producing new technologies, but it also means that there is a potential incentive problem -- in the absence of well-developed intellectual property rights, there will be no incentive to develop new technologies. Patent protection is an example of how we set up rules to give incentives for technology development.

So in our model, we assume that the inventor of a new intermediate good gets a perpetual monopoly on the right to produce it in perpetuity.

The cost of producing a new invention is assumed to be a constant, \( \eta \).
Note two ways in which this assumption might be made more realistic. First, it might be better to assume that there is some stochastic element in the process of inventing stuff. Second, it might be reasonable to argue that there aren’t constant returns to scale in invention -- if you double R&D spending you don’t double results. (See Aghion and Howitt for a model that fixes at least problem number 1 -- maybe number 2 also).

What is the value of having invented a new good?

$P_j$ is the price. We assume that the cost of producing one unit of the intermediate good is one. so

$$V(t) = \int_{t}^{\infty} (P_j - 1)X_j e^{r(t-v)} dv$$

where we are assuming that the interest rate is constant -- see Barro and Sala-i-Martin for the demonstration that it is.

The monopolist chooses $X$ and $P$ to maximize profit, subject to the demand curve derived earlier.

$$\text{profit} = (P_j - 1)X_j = (P_j - 1)L\left(\frac{\alpha}{P_j}\right)^{\frac{1}{1-\alpha}} = (P_j - 1)L\alpha^{\frac{1}{1-\alpha}}P_j^{\frac{1}{\alpha}}$$

$$\frac{d \text{profit}}{d P_j} = L\alpha^{\frac{1}{1-\alpha}}\left[\frac{-1}{P_j^{\frac{1}{\alpha}}} + \frac{-1}{1-\alpha}P_j^{\frac{2-\alpha}{\alpha}}\right] = 0$$

which solves to $P_j = 1/\alpha$

so $X_j = L \alpha^{2/(1-\alpha)}$

Note that the price is the same for all $j$, and similarly $X_j$ will be the same for all $j$ (but only for those that exist at a given time).

$$V(t) = \int_{t}^{\infty} (P_j - 1)X_j e^{r(t-v)} dv = \left(\frac{1}{\alpha} - 1\right)L\alpha^{\frac{2}{1-\alpha}}\int_{t}^{\infty} e^{r(t-v)} dv$$
We assume that there is free entry (and thus zero profits) in getting into the business of being an inventor/monopolist. That is, once you get an invention, it will earn you monopoly rents -- but the PDV of these rents will be exactly equal to the cost of getting into the business, that is, to $\eta$. So setting $V(t)=\eta$, we solve to get:

$$
V(t) = \left( \frac{1}{\alpha} - 1 \right) L \alpha^{\frac{2}{\alpha}} \left( \frac{1}{r} \right)
$$

So the cost of invention (and the other technological parameters) pin down the interest rate.

What is going on here? There is no capital, so the only "investment" that you can make is creating a new intermediate good; the value of this is determined by the technology of inventing.

Note that we are assuming that some inventing is being done -- another way to put this point is that there is infinitely elastic demand for borrowing at the interest rate given above. Of course, if the interest rate is higher than this, then no inventing will be being done, and so this equation for the interest rate will not hold.

Note that smaller $\eta$ (inventing is easier) implies rate of return is higher.

**Households**

We assume that households have CRRA preferences, with $\sigma$ at the coefficient of relative risk aversion, and $\Theta$ as the time discount rate. So we know the rate of consumption growth

$$
\frac{\dot{c}}{c} = \frac{1}{\sigma} (r - \Theta)
$$

so given $r$,

$$
\frac{\dot{c}}{c} = \frac{1}{\sigma} \left[ \left( \frac{L}{\eta} \right) \left( \frac{1 - \alpha}{\alpha} \right) \alpha^{\frac{2}{\alpha}} \theta - \theta \right]
$$
Note that if this led to negative consumption growth, there would be a problem. Since there is no capital stock to run down, consumption can't fall. So we assume the parameters are such that consumption growth is positive.

Aggregate output is

\[ Y = \sum_{i} Y_i = \sum_{i} L_i^{1-a} N X_i^\alpha \]

[What is going on with this next line??? fix it.]

\[ = N \sum_{i} L_i^{1-a} \left( L \alpha^a \right)^{\alpha} = N \left( L \alpha^{\frac{2\alpha}{1-a}} \right) \]

so \( \frac{\dot{Y}}{Y} = \frac{\dot{N}}{N} \)

note that we are fudging and making \( N \) continuous. We could have done this from the beginning by writing the firm-level production function as

\[ Y_i = L_i^{1-a} \int_{0}^{N} \left( X(j) \right)^{\alpha} dj \]

Now we just need to show that the growth rates of \( N \) and \( Y \) have to be equal to the growth rate of \( c \).

From the economy-wide budget constraint we know that consumption has to be equal to output less spending on intermediate goods and spending on new inventions:

\[ C = Y - NX - \eta \dot{N} \]

divide all of the terms in this by \( Y \)

\[ \frac{C}{Y} = 1 - X \left( \frac{N}{Y} \right) - \eta \frac{\dot{N}}{Y} = 1 - X \left( \frac{N}{Y} \right) \cdot \eta \left( \frac{N}{Y} \right) \]

where for the last step we have multiplied and divided by \( N \), and substituted \( \frac{\dot{Y}}{Y} = \frac{\dot{N}}{N} \).
We know from above that $N/Y$ is constant; we also know that $X$ (the quantity of each intermediate good used) is constant.

\[
\frac{\dot{Y}}{Y} = \frac{C}{Y} \cdot x\left(\frac{N}{Y}\right) + I
\]

So suppose that the growth rate of output is not equal to the growth rate of consumption at one point in time, it will diverge from it permanently (ie if output growth is higher than consumption growth, then $C/Y$ will fall, and output growth will rise). Eventually, this will violate some implicit transversality condition.

Extensions:

note that we have size effects, which are not present empirically. This suggests two things:

-- 1) we should think of a model where technology crosses borders

2) for the world, might be right that there are size effects, but it depends on how cost of new technologies changes as you try to develop them more quickly. (ie question of constant returns to R&D, raised above).

Note also that we haven't said anything about optimality -- it turns out that there is too little invention (and the prices of intermediate goods are too high) compared to the social planner's outcome.

[End of Material Not Covered when Howitt is Teaching]

Malthus: do simple graphical version.

Lucas' version of the Malthus/Ricardo population model

Simple demographic structure. Individuals live in dynasties. There is one economically active period, in which individuals work, consume, and raise children. For simple version of the model, we assume that all members of a generation are identical.

$c_t = \text{consumption per person}$
\( n_t = \text{children per person} \)

Production:

\[
\begin{align*}
L &= \text{quantity of land (not labor!)} \\
N &= \text{number of families = number of workers}
\end{align*}
\]

in the simple version of the model there is no capital. Land is the only productive resource.

production is CRS: \( Y = F(L,N) = A L^\alpha N^{1-\alpha} \)

define \( x = \frac{L}{N} \) (land per worker)

\( y = \text{output per worker} \)

\( y = A x^\alpha \)

We assume that land is not owned privately, so it earns no return. Individuals earn their average product -- that is, \( y \). (Can think of this as a pre-enclosure grazing economy, or a hunting and gathering society, or fishing, or any other common resource. Note -- this characterization of people in agriculture earning their average (not marginal) product crops up in a lot of development literature. Later in the paper Lucas considers the case where land is privately owned).

children cost \( k \) each to raise (no worry about integer constraints). [Think about in a more advanced model: probably not a good idea to make the cost of children fixed in terms of goods. The real cost of children is in _time_, which means that as the wage rises, the cost of children goes up].

So the budget constraint is

\[ c + kn = y \]

What about utility:

The general formulation is to give people utility over their own consumption, the number of kids, and utility per kid. We specialize this to be additively separable.

\[
u_t = W(c_t, n_t, u_{t+1}) = (1 - \beta) \ln(c_t) + \eta \ln(n_t) + \beta u_{t+1}\]

Notice that in this version of the model there is nothing that the parent can do to affect the
utility of the kids. (That is, the parent can't give the child land (which is not owned) or capital (which does not exist). The fact that the kids utility enters the parent's utility in a separable fashion means that it does not affect the marginal utility of anything else, and so it will not have any effect on the parent's decisions. So it is not really doing anything in this version of the model.

We can maximize utility subject to the budget constraint, and get the following first order condition:

\[
\frac{n_t}{c_t} = \frac{\eta}{(1 - \beta)k}
\]

We can actually learn a lot just from this. We know that in the steady state, \(x\) (which is land per worker) will be constant. So \(N\) must be constant, and so in steady state, \(n=1\). So we can solve for the steady state level of consumption directly:

\[
c_{ss} = \frac{(1 - \beta)k}{\eta}
\]

Interesting things to note about this: steady state consumption doesn't depend on \(L, A, \alpha\), etc. It just depends on preference parameters and the cost of kids. The more that people value children, and the less that people value their own consumption, the lower is steady state consumption. The more expensive are children, the higher is steady state consumption.

What about the model's dynamics?

the evolution of land per worker \((x)\) is given by

\[
x_{t+1} = \frac{L}{N_{t+1}} = \frac{L}{n_t N_t} = \frac{n_t}{x_t}
\]

Combining the FOC from utility maximization with the budget constraint, we can solve for \(n_t\) in terms of \(y_t\), and then substitute in the production function

\[
n_t = \frac{y_t}{k} \left( \frac{\eta}{1 - \beta + \eta} \right) = \frac{A x_t^\alpha}{k} \left( \frac{\eta}{1 - \beta + \eta} \right)
\]

Substituting this into the equation for the evolution of \(x\):
We can graph \( x_{t+1} \) against \( x_t \), and include a 45 degree line, to see that this leads to a single stable equilibrium.

We can also use this equation to solve for the steady state values of \( x \), and therefore of \( y \).

\[
x_{ss} = \left[ \frac{k}{A} \left( \frac{1 - \beta + \eta}{\eta} \right) \right]^{\frac{1}{\alpha}}
\]

\[
y_{ss} = k \left( \frac{1 - \beta + \eta}{\eta} \right)
\]

Comparing this with steady state consumption, we can see that the steady state ratio of consumption to output is \( (1-\beta)/(1-\beta+\eta) \), which corresponds to their weight in the relevant part of the cobb-douglass utility function.

Note that the equation for steady state output says that it is higher, the more expensive are kids.

So, like consumption, output per worker does not depend of the level of technology (A), or the amount of land. What about the total number of people, \( N \)?

Using the equation for \( x_{ss} \)

\[
N_{ss} = L A^{\frac{1}{\alpha}} \left[ k \left( \frac{1 - \beta + \eta}{\eta} \right) \right]^{\frac{1}{\alpha}}
\]

So the number of people is proportional to the amount of land -- no big surprise. Further, if there is better technology, you get more people per unit of land (that is, higher population density), but you get no higher per capita output or consumption. This is the Malthusian deal: population grows to run resources down.

In the rest of the paper, Lucas goes on to consider more sophisticated versions of the model -- with capital and private property, for example.
Does this characterize the world?

Lucas argues that it is a good characterization of the world until recently: Prior to roughly 1700, countries seemed to be in Malthusian steady states. Income per capita was roughly constant at the subsistence level. Better technologies meant more people, but not higher living standards.

Then around 1750 (in the west), something changed: income per capita started rising, and the gains in income did not immediately result in higher population.

### Phases of Economic Growth in Europe

<table>
<thead>
<tr>
<th>Phase</th>
<th>Population</th>
<th>GDP per capita</th>
<th>GDP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agrarianism</td>
<td>0.1</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td>500-1500</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Advancing Agrarianism</td>
<td>0.2</td>
<td>0.1</td>
<td>0.3</td>
</tr>
<tr>
<td>1500-1700</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Merchant Capitalism</td>
<td>0.4</td>
<td>0.2</td>
<td>0.6</td>
</tr>
<tr>
<td>1700-1820</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Capitalism</td>
<td>0.9</td>
<td>1.6</td>
<td>2.5</td>
</tr>
<tr>
<td>1820-</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Source: Angus Maddison, Phases of Capitalist Development

Explaining what changed is really hard. It is easy to have a Malthusian model with stable population -- but once you get away from the resource constraint, it is very hard to have a model that produces constant population....

Notice that in the Malthusian regime, we can read the progress of technology directly from the population density. So the more than doubling of population in Europe over the period 500-1500 means that there was a lot of technological improvement, even though living standards did not change.
Also explains why in, say, 1950, China had the same income per capita as Africa, but much higher technology -- it also had less land per person.

Kremer’s Model of Population and Technology

We now turn to a related model that integrates population and technology change.

The setup of the model is simple. Output is produced with land and labor -- no capital (capital could easily be incorporated, and it would not make a difference as long as the production function was such that there was decreasing return to labor and capital while land was held fixed -- in other words as long as there was constant returns to land, labor, and capital together).

\[ Y = R^\alpha [A L]^{1-\alpha} \]

It is assumed that the growth rate of A (knowledge) is proportional to the size of the population. The argument here is that people are equally likely to come up with new productive ideas, so that the more people you have, the more ideas you will come up with. [it is also assumed that the productivity of each new idea is proportional to the current level of productivity]

\[ \dot{A} / A = BL \]

finally, there is a Malthusian model of population. Population adjusts to keep output per person at the subsistence level (this adjustment is not modeled, but you can think of a Lucas-style model in the background).

\[ Y / L = y = \bar{y} \]

Given the assumption that income is at subsistence and the production function, we can solve for the level of population as a function of the state of technology and the quantity of land:

\[ \bar{y}L = R^\alpha [AL]^{1-\alpha} \]

\[ L = \left( \frac{1}{y} \right)^{1/\alpha} A^{1-\alpha} R \]
So once again population is proportional to land, and we can read the level of technology off from the level of population density.

Since $\alpha$ and $R$ are constant, the growth rate of population is related to the growth rate of technology:

$$\frac{\dot{L}}{L} = \frac{1 - \alpha}{\alpha} \frac{\dot{A}}{A} = \frac{1 - \alpha}{\alpha} BL$$

So the growth rate of population is proportional to the level of population. That is, the growth rate of population is increasing over time.

This seems like a pretty silly prediction, but when he looked at data it fit pretty well. Specifically, he plots population against the growth rate of population from 1 million BC to the present, and it fits as an almost straight line (see figure in Romer book). It only falls apart in the last few decades.

Of course we know from the discussion of Lucas above that the Malthusian model stops fitting in the developed countries a while ago...

He also cites evidence about the relative growth rates of different isolated populations...
Problems

1) Consider the Constant Elasticity of Substitution (CES) production function:

\[ Y = [aK^{-\beta} + L^{-\beta}]^{-(1/\beta)} \] where \( \beta > -1 \)

A. Show that this production function has constant returns to scale.

B. Calculate the marginal products of capital and labor.

C. Calculate the fraction of total output that goes to capital. For what values of \( \beta \) is it the case that as the capital/labor ratio rises, the fraction of total output that goes to capital rises? For what values does it fall? For what values is it constant?

2) Output per worker is 5 times as high in country A as in country B. Assume that the countries have the same Cobb-Douglass production function and that capital's share in income in each country is 1/3. By what factor does the MPK differ between the two countries? What about if capital's share in each country is 2/3?

3) In a country, the production function is \( y = k^{1/2} \), where \( y \) is output per worker and \( k \) is capital per worker. Suppose that in 1990, \( k \) is equal to 400. The fraction of output saved is 50%. The depreciation rate is 5%. Assume that there is no population growth or technological change.

A. Is the country at its steady state level of output per capita, above the steady state, or below the steady state? Show how you reached your conclusion.

B. Now suppose that there is a large immigration to the country. The population of the country quadruples. The new immigrants do not bring any capital with them. Following the immigration, is output per capita above, below, or at its steady state level?

4) Consider a Solow model, where the production function is

\[ Y = A K^{\alpha} L^{1-\alpha} \]

Population grows at rate \( n \) and capital depreciates at rate \( \delta \). There is no technological change.
There are two countries (both of which are closed to capital flows) that are both at their steady states. They have the same levels of $y$, $n$, and $\delta$. The value of $A$ in country 2 is twice as large as in country 1. What is the ratio of their saving rates?

5) (old final exam question) Consider a country with production function (in per capita terms) $y=k^\alpha$ ($0<\alpha<1$). Population grows at rate $n$ and capital depreciates at rate $\delta$. There is no technological change. Consumption is equal to a constant fraction, $c$, of output. In addition, every period a payment in the amount $p$ per capita must be made to the foreign power which provides protection for this country. All output that is not consumed or paid to the foreign country is invested.

A. Write down the differential equation governing the evolution of the per capita stock of capital in this country.

B. Draw the usual Solow model diagram for this country. Is there more than one equilibrium level of the capital stock and of output? If so, identify all of the equilibria. Indicate on your diagram which equilibrium the economy will move to depending on its initial stock of capital per worker.

6) Consider two countries, A and B, which are described by the Solow model. In both countries, the production function (in per capita terms) is $y=k^{.5}$, and the rate of depreciation ($\delta$) is .02. There is also no technological change. In A, the saving rate ($s$) is .2 and the population growth rate ($n$) is zero. In B, the saving rate is .4 and the population growth rate is .02.

Both countries are initially observed to have income per capita of 5 at time zero. Draw a graph with time on the horizontal axis and income per capita on the vertical axis showing how the level of income per capita in the two countries will evolve over time. Explain.

7) A country described by the Solow model has a production function $y=k^{.5}$, and some rates of population growth ($n$) and depreciation ($\delta$). There is no technological change. Currently, the saving rate in the country is .6, and the country is at the steady state level of output per capita. Two courses of action are proposed: One is to lower the saving rate to .5, and the other is to lower the saving rate to .4. Graph the paths of consumption per capita over time following the introduction of the new policy, and show how these levels of consumption compare to the level of consumption per capita in the old steady state. Explain what is going on.

8) Consider a Solow model with positive rates of population growth, depreciation, and technological change. Imagine that a country is in steady state, and that suddenly its rate of technological change increases. Describe how output per efficiency unit and the growth rate of output per capita evolve.

9) Assume a Solow model in which two countries have the same savings rates and the the
same values of \( n, \delta, \text{ and } g \), but differ in the state of technology.

\[
y_m = A_m k_m^\alpha \quad \quad y_n = A_n k_n^\alpha
\]

Output in country \( m \) is five time higher than in country \( n \). Both countries are in steady state.

A. By what factor do the two countries' capital stocks differ?
B. By what factor do the two countries' values of \( A \) differ?
C. By what factor do the two countries' \( \text{MPK} \) differ?

10) A country has the following production function:

\[
Y = A(bK)^\alpha(cL)^{1-\alpha} \quad \quad 0<\alpha<1
\]

the growth rate of \( A \) is \( g_A \), the growth rate of \( b \) is \( g_b \), and the growth rate of \( c \) is \( g_c \). Population growth is zero. Calculate the growth rate of output per capita in the steady state of a Solow model.

11) Consider a Solow model in which the rate of growth of population is endogenous. Specifically, the rate of population growth is \( n_h \) when \( y< \) and is \( n_l \) when \( y \geq \). \( n_l < n_h \). The rest of the model is standard. Production is Cobb-Douglas: \( y = k^\alpha \). Capital depreciates at some rate \( \delta \). There is no technological change. The saving rate is exogenous.

A. Draw pictures showing the different possible configurations of equilibria (ie a single, globally stable equilibrium vs multiple equilibria) in the model.

B. For what range of values of the saving rate will it be the case that the model displays multiple equilibria? For what values will there be a single equilibrium?

12) Consider two countries described by the Solow model.

They both have the same saving rates, population growth rates, and depreciation rates. There is no technological change. In each country the production function is (in per capita terms):

\[
y = k^\alpha \quad 0<\alpha<1
\]

In country 1 there is a proportional tax at rate \( \tau \). The proceeds of the tax are thrown into the sea, and do not have any effect on the economy. Saving is done out of post-tax income.

In country 2, there is a constant per-capita tax (ie a head tax) of \( \Psi \) per person. As in country one, the proceeds of the tax are thrown into the sea. Saving is done out of post-tax income.
The per-person tax in country 2, $\Psi$, is set to be equal to taxes collected per person in country 1 when country 1 is in steady state.

A. Solve for $\Psi$.

B. Describe the dynamics of the two economies (how they evolve from different initial levels of capital, etc.). Show that there different possible configurations of steady states in country two relative to country one.

C. Show how the specific configuration of steady states in country 2 relative to country one depends on the parameters. Derive the conditions under which the different possible cases will appear.

13) Consider a Solow model in which the production function (in per capita terms) is

$$y = k^{1/3} h^{1/3},$$

where $k$ is physical capital per worker and $h$ is human capital per worker. Physical capital is accumulated according to

$$\dot{k} = s_k y - (n+\delta)k$$

and human capital is accumulated according to

$$\dot{h} = s_h y - (n+\delta)h$$

where $n$ is population growth, $\delta$ is depreciation, and $s_k$ and $s_h$ are the rates of saving in physical and human capital.

Two countries have the same levels of $s_k$, but differ by factor of two in their levels of $s_h$. By what factor will their steady state levels of output differ?
14) Country one is exactly described by the standard Solow model (with no technological progress):

\[ y = k^\alpha \]

\[ \dot{k} = sy - (n + \delta)k \]

The values of the parameters are: \( \alpha = .5 \), \( (n+\delta) = .1 \), \( s = .1 \)

Country 2 is the same as country 1, except that the saving rate is a function of the capital stock:

\[ s = s_0 \left( \frac{I}{I+k} \right) \]

Where \( s_0 = .2 \).

A. Show that the two countries have the same steady state levels of output. Intuitively, which country should move more rapidly toward its steady state?

B. Linearize around the steady state to get an expression for output growth of the form:

\[ \frac{\dot{y}}{y} = -\gamma(\ln(y) - \ln(y^*)) \]

How does the value of \( \gamma \) compare in the two cases?

15) \( s =.2 \); \( n=.02 \); \( g=.02 \); \( \delta=.03 \); \( \alpha = 1/3 \). Use the linearized form of the simple Solow model to calculate how quickly output per capita should be growing in a country with income per efficiency unit equal to half its steady state level. Do the same for a country with income per efficiency unit equal to one quarter of its steady state level. Calculate the time that the country will take to get from one quarter its steady state level of output per efficiency unit to one half its steady state level.

To test the quality of this linearization: use a calculator, computer, etc. to figure out the actual rate of output growth for the two countries examined in question 1. Finally (this requires a little more work), calculate the time that the country will take to get from one quarter to one half the steady state level of income per efficiency unit.

16) Consider a world in which there are two countries with equal sized labor forces. The
countries have the same production function. In per-worker terms:

\[ y = k^\alpha \quad 0 < \alpha < 1 \]

there is no technological progress. The depreciation rate is zero. In each country, factors are paid their marginal products. The growth rates of the labor force in the two countries are the same and are greater than zero:

\[ n_1 = n_2 = n > 0 \]

Capital flows freely between the two countries to equalize the marginal product of capital.

In each country, savings takes place only out of capital income. In other words, all wages earned by residents of the country are consumed. Of payments to capital owned by people who live in country i, a fraction \( s_i \) is saved and \( (1-s_i) \) is consumed. Notice that this saving rate applies to income paid to capital owned by people who live in country i, whether or not that capital is located in country i.

The saving rate out of capital income is higher for people in country 2 than country 1, in other words,

\[ s_2 > s_1 \]

Define \( a_1 \) and \( a_2 \) as the levels of assets per worker in countries 1 and 2, and \( k_1 \) and \( k_2 \) as the levels of capital per worker in each country.

Solve for the steady state levels of capital per worker and assets per worker in each country.

17) Consider a Ramsey model, with depreciation \( \delta \), population growth \( n \), time discount rate \( \Theta \), and production function (in per capita terms) \( f(k) = k^{\frac{5}{2}} \). Solve for the steady state level of consumption per capita in terms of the three parameters.

17.5) (Midterm exam, 2003) Consider a Ramsey model with a social planner, where instead of time going to infinity, it is know from time zero that the world will end at time T. You should assume that the capital stock at time zero is well below the steady state that would apply if time were infinite. You should also assume that T is a long way in the future. Draw the usual diagram for the Ramsey model, including the stable arm that would be appropriate if time went to infinity. On this diagram, show the paths of consumption and capital stock. Also draw graphs of consumption and capital with time along the horizontal axis. Explain how you got your results.
18) Consider two economies described by the Ramsey model, which are the same in every respect (i.e., population growth rate; time discount rate, etc.) except their discount rates: People in country A discount the future more than people in country B. Assume that both countries start off the same capital stock $k_0$ which is below either country’s steady state level. Which country will have higher initial consumption? Is it possible for the two countries' stable arms to cross?

18.5) [midterm exam, 2005] Consider a Ramsey model in which the social planner weighs future consumption according to the number of people who will be alive in the following manner. The social planner wants to maximize

$$V = \int_0^\infty U(c(t))N(t)\pi e^{-\theta t} dt$$

where $0 < \pi < 1$. Population grows exogenously at rate $n > 0$. The rest of the problem is standard. There is no technological progress. The depreciation rate is zero. The production function is

$$y = k^\alpha$$

and the differential equation for capital stock is

$$\dot{k} = f(k) - c - nk.$$ 

A) What assumption regarding the values of $n$, $\pi$, and $\theta$ is required to ensure that the social planner’s utility is finite?

B) Making this assumption, solve for the steady state level of capital per worker.

19) (midterm, 2001) Consider two economies described by the Ramsey model. The economies have the same production functions and rates of population growth. Both of them are also closed to capital flows from the rest of the world.

The countries differ in the preferences of their social planners, that is, in $\theta$, the time discount rate, and $\sigma$, the coefficient of relative risk aversion (both social planners have CRRA utility.) We know that $\theta_1 > \theta_2$. We do not know how their values of $\sigma$ compare (but we will figure it out.)
The time-zero values of capital per worker, $k_0$, are equal in the two countries. Also, by coincidence, the optimal values of time-zero consumption chosen by the social planners, $c_0$, are also equal.

Figure out which country has a higher value of $\sigma$. Also draw a graph with time on the horizontal axis and consumption on the vertical axis, and show how the time paths of consumption in the two countries compare.

19.5) [midterm exam, 2005] A country has production function

$$Y = Ak$$

Time is continuous. Population is constant. There is some depreciation rate $\delta > 0$. Consumption decisions are made by a social planner who has instantaneous utility of the CARA form:

$$u(c) = -\frac{1}{\alpha} e^{-\alpha c}$$

The social planner has a time discount rate $\theta > 0$. You should assume that $A - \delta > \theta$.

Reminder: the continuous time FOC for an individual with CARA utility facing an interest rate $r$ is

$$\dot{c} = \frac{1}{\alpha} (r - \theta)$$

A. Write down the differential equations for $\dot{k}$ and $\dot{c}$ (not $\dot{k}/k$ or $\dot{c}/c$).

B. Draw a diagram with $k$ on the horizontal axis and $c$ on the vertical axis. Draw arrows indicating the dynamics of $c$ and $k$.

C. Suppose that there is some initial capital stock $k(0)$. Add to the diagram from part (B) some representative paths the economy would follow based on different choices of initial consumption. Indicate which path the social planner will choose.

D. Suppose that there are two countries run by social planners with the same preferences. The two countries have the same value of $A$. The countries differ only in their initial capital stocks. Suppose that at time zero, $\frac{y_1}{y_2} = 2$. Toward what value will the ratio of incomes in the two countries asymptote?
E. Solve for initial consumption, \( c(0) \), as a function of initial capital stock, \( k(0) \).

20) Consider a Ramsey model with government spending. The differential equation for the evolution of capital is

\[ k = f(k) - c - nk - G \]

There are two levels of government spending, \( G_h > G_l \). Spending alternates between these two levels in a fixed cycle. Use the phase diagram to describe how consumption and capital will behave once the economy has been in this pattern for a long time. Draw time series pictures of how the interest rate behaves over the “political cycle.” Consider the limiting cases as the period of alternation becomes very short or as it becomes very long. (hint: thinking about very long periods of alternation is the easiest way to approach the problem).

21) (final, 2001) Consider a typical Ramsey model in which the social planner maximizes the discounted integral of future per-capita utility. Population grows at rate \( n \). The time discount rate is \( \theta \). The equation for the accumulation of capital is

\[ \dot{k} = f(k) - c - nk \]

The production function is

\[ y = A k^\alpha \]

Prior to time \( t \), the value of \( A \) is constant and equal to \( A_1 \), and the economy is in steady state. At time \( t \), it is announced that at some future date, time \( s \), the value of \( A \) will jump from \( A_1 \) to \( A_2 \), where \( A_1 > A_2 \). After this, \( A \) will remain constant.

Consider the different possible relations between the position of the stable arm associated with the new steady state and the initial steady state (there are three possibilities.) For each of these cases, draw a diagram with time on the horizontal axis and the marginal product of capital on the vertical axis, and show how MPK will change over time. In each case, show where \( r \) is rising, falling, jumping, etc. Also show how the level of \( r \) in the new steady state compares to the initial value.

22) [core exam, 2003] In a certain country, consumption and investment decisions are made by a social planner who maximizes
\[
\int_{0}^{\infty} \ln(c) e^{-\theta t} \, dt
\]

where \( c \) is consumption per capita, and \( \theta > 0 \) is the time discount rate. The population (which is equal to the labor force) grows exogenously at rate \( n > 0 \).

There are two kinds of capital used in production. The quantities per worker are denoted \( k_1 \) and \( k_2 \). Once capital has been created, it cannot be turned back into output. Similarly, one type of capital cannot be turned into the other. Call \( i_1 \) the per worker quantity of investment in type 1 capital and \( i_2 \) the per worker quantity of investment in type 2 capital.

The social planner’s budget constraint is

\[ y = c + i_1 + i_2 \]

The per-worker production function is

\[ y = k_1^\alpha k_2^\alpha \quad \alpha < \frac{1}{2} \]

The deprecation rate is zero.

A. Solve for the steady state levels of \( c, k_1, \) and \( k_2 \). [Note, there are some slightly ugly algebraic expressions that crop up here and in part C. You do not have to simplify too much, as long as I can see what you doing.]

Now, suppose that the economy is in steady state when suddenly an asteroid collides with the earth, destroying half of the type 2 capital. None of the type 1 capital is destroyed.

B. Draw a picture with the quantity of \( k_1 \) on the horizontal axis and the quantity of \( k_2 \) on the vertical axis. Draw a point indicating the steady state. Now sketch on the figure the time path of \((k_1, k_2)\) following the collision with the asteroid. Also describe (in words) how investment behaves during this transition.

C. Solve for the growth rate of consumption immediately following the collision with the asteroid.

23) A country is initially closed to the international capital market. Production and consumption are described by the Ramsey model. The interest rate is \( r^a \) (a for autarky).
The world interest rate, $r^*$, is greater than the closed-economy interest rate: $r^* > r^a$. The economy is small, so that once it opens to the world economy it will be a small open economy.

Assume that the opening to the world economy is a surprise. Prior to it, the economy is in steady state.

Describe the time paths of consumption, output, and net foreign assets at the time of the opening and subsequently. Be sure to indicate which variables do or do not jump at the time that the economy is opened up. [Note: figuring out what happens to consumption initially may not be possible, but say what you can about it]

23.5) [midterm exam, 2004] Consider a simple OLG model in which people supply one unit of labor in each period of their lives. They are born and die with zero assets. Assume production is Cobb-Douglas with $\alpha = \frac{1}{2}$. Also assume that capital fully depreciates after it is used, so that old people consume only the marginal product of their capital, not the capital itself. People have log utility with a time discount rate of zero. Population is constant.

Solve for the steady state level of output per worker. (Remember, both old and young people work).

23.75) [final exam, 2005] Consider the following OLG economy. There is no population growth or technological change. Individuals work only in the first period of life. Individuals have Leontief lifetime utility of the form

$$u = \min(c_1, c_2)$$

Firms have per-worker production function

$$y = k^\alpha$$

Where $\alpha < \frac{1}{2}$. Factors are paid their marginal products. The depreciation rate is $\delta > 0$. 

Solve for the steady state level of capital per worker.

24) (old midterm question) Consider an open economy populated by OLG people. There is no depreciation, time discounting, technological change, or population growth.

Utility is: $U = \log(c_1) + \log(c_2)$
The world interest rate, \( r^* \) is taken as exogenous by the country. People can either borrow or lend to the rest of world at interest rate \( r^* \).

The production function is \( f(k) = k^\alpha \)

Calculate the steady state levels of capital, the wage, saving, and wealth in the country as a function of \( r^* \). What determines whether, in steady state, the country has positive or negative net exports?

24.5) [midterm exam, 2005] Consider two economies described by the standard OLG model. There is a standard production function and no population growth. In country 1, people have time discount rate \( \theta_1 \) and log utility. In country 2, people have time discount rate \( \theta_2 \) and CRRA utility with coefficient of relative risk aversion \( \sigma > 1 \). The two countries have the same steady state levels of capital and output per person.

Draw a diagram with \( k_t \) on the horizontal axis, \( k_{t+1} \) on the vertical axis, and a 45-degree line. On this diagram, show how the functions \( k_{t+1} = \phi(k_t) \) for the two countries relate to each other. Which country will move more rapidly toward its steady state. Explain fully how you know what the picture looks like.

25) [core exam, 2001] Consider the following Overlapping Generations economy: People live for two periods. The lifetime utility of an individual born at time \( t \) is given by

\[
U_t = \ln(c_{1,t} - x) + \ln(c_{2,t+1}) \quad x > 0
\]

where \( c_{1,t} \) is consumption in the first period of life of a person born at time \( t \), and \( c_{2,t+1} \) is consumption of the same person in the second period of life. \( x \) is an exogenous constant.

Other than having \( x \) in the utility function in the first period of life, the model is completely standard. People work in the first period of life, inelastically supplying one unit of labor. They do not work in the second period of life. The savings of the current elderly generation make up the capital stock used in production. Population size is constant. Factors are paid their marginal products. The rate of depreciation is zero. The production function, in per worker terms, is

\[
y = k^\alpha
\]

A. Draw a diagram showing how capital evolves from period to period (i.e. a diagram with \( k_t \) on the horizontal axis and \( k_{t+1} \) on the vertical axis.) Indicate any possible steady states and
discuss their stability. Show how the number of steady states and their stability is affected by the size of x.

B. Now suppose that the value of x is endogenous. In particular, assume that x at time t (which applies only to the young people at time t) is based on the consumption of old people at time t:

\[ x_t = c_{2, t} \]

Discuss the conditions under which a stable steady state will exist in this economy.

26) Consider an OLG model in which there is capital mobility. There are two countries in the world, with equal populations. There is no population growth, technological progress, or depreciation. The production functions in the two countries are the same: \( Y = K^{0.5}L^{0.5} \). In each country, people work in the first period of life only, and consume in both the first and second periods of life. Their utility functions are:

Country 1: \[ U = (1-\gamma) \ln(c_y) + \gamma \ln(c_o) \]

Country 2: \[ U = (1-\beta) \ln(c_y) + \beta \ln(c_o) \]

where \( c_y \) is consumption when young, \( c_o \) is consumption when old, and \( 1>\gamma>\beta>0 \).

A. Solve for the steady-state level of GDP per worker in each country.

B. Solve for the steady-state level of GNP per worker in each country.

27) Consider the following variation on the OLG model presented in class. Individuals live for two periods. In the first period of life they work, supplying one unit of labor. They do not do any consumption during the first period of life. In the second period of life, they do not work, and they consume their savings from the first period along with any accrued interest. The economy is closed.

The production function, in per-worker terms, is \( y=k^{\alpha} \). The depreciation rate is zero. The population grows at rate \( n \) -- that is, each generation is \((1+n)\) times as big as the one that came before.

A. Solve for the steady-state level of capital per worker.

B. Assume that \( \alpha \) is equal to one-third. For what value of \( n \) will this economy be at the Golden Rule level of the capital stock?
27.5) [midterm exam, 2004] Consider an OLG model in which people live for two periods. In the first period, they work, raise children, and consume. In the second period, they consume. Their utility functions are

\[ V = \ln(c_1) + \ln(c_2) + \ln(n) \]

where \( n \) is the number of children. The price of children (in terms of the consumption good) is one.

The production function is \( y = k^\alpha \), where \( k \) is capital per worker and \( y \) is output per worker. The depreciation rate is zero. Factors are paid their marginal products.

**Note:** the number of children per worker in this model is \( n \), not \( (1+n) \), which is how we usually denote it.

A) [15 points] Solve for the steady state level of output.

B) [15 points] Assume \( \alpha = \frac{1}{2} \). Consider two countries: one is described by the model above; in the other one, \( n \) is fixed at 1/6. Describe (and graph) the difference equation relating \( k_{t+1} \) to \( k_t \) in both countries. If both countries started off at half of their steady state level of output per worker, which would move more rapidly to the steady state? Explain why.

28) Consider a variation of the OLG model in which people may die at the end of the first period. At the beginning of the second period of their lives, people die with probability \( (1-p) \) and live with probability \( p \). The do not know during the first period of their lives whether they will die or not.

There is no time discounting. Their expected utility (which they maximize) is

\[ E(U) = \ln(c_1) + p\ln(c_2) \]

where \( c_2 \) is what they will consume in period 2, if they live that long.

The production function is \( y=k^\alpha \), and there is no population growth.

Assume that when people die their wealth is distributed among the remaining members of their generation. Assume that there are a sufficiently large number of people in each generation so that there is no uncertainty about the size of bequests that survivors will receive. Also assume that interest is earned by the capital belonging to the deceased before this capital is divided up among the survivors.
A. Write down the single period and lifetime budget constraints of an individual. Call the amount that she receives as a bequest b.

B. Solve for the individual's optimal saving in period 1 as a function of b, r, and w.

C. Solve for b as a function of the amount of capital in the second period.

D. Put everything together into a difference equation for K.

E. How does a decrease in p (that is, an increase in the probability of premature death) affect the steady-state capital stock?

29) (Core exam, 2001-02) Consider the following OLG model. People live for two periods. They are born and die with zero assets. They work only in the first period of their lives. The capital stock in any period is composed of the savings of the older generation.

People have the following lifetime utility functions:

\[ U = \ln(c_1) + \ln(c_2) \]

The production function is

\[ y = k^\alpha, \]

where y is output per worker and k is capital per worker. Workers are paid their marginal products.

Population grows at rate n, so that the number of young in period t+1 is \((1+n)\) times the number of young people in period t.

Capital fully depreciates after it is used in production. This means that the elderly consume the earnings from the capital they own, but not the capital itself.

The economy is in steady state.

A) Solve for the steady state level of capital per worker.

B) Solve for the steady state levels of first and second period consumption.

C) Solve for the steady state level of lifetime utility.

D) Assume that \( \alpha = \frac{1}{2} \). Is steady state lifetime utility higher in a country where population growth is positive (i.e. \( n>0 \)) or negative (i.e. \( n<0 \))? Indicate how you know this.
29.5) (Midterm exam, 2003) Consider an OLG model in which people live for two periods. They work in the first period and consume in both periods. The size of the population is constant. People have log utility with a time discount rate of zero.

The production function is

\[ Y = K^\alpha (eL)^{1-\alpha}, \]

where \( e \) is a measure of the number of efficiency units of labor per worker. Define capital and output per efficiency unit of labor as follows:

\[ k_t = K_t / e_t L \]
\[ y_t = Y_t / e_t L \]

Technological progress takes the following form. The value of \( e \) does not change from an odd period to an even period. But from an even period to an odd period, \( e \) increases by a factor of \((1+\gamma)\). So, for example,

\[ e_1 = e_0 (1 + \gamma) \]
\[ e_2 = e_1 \]
\[ e_3 = e_2 (1 + \gamma) \quad , \text{etc.} \]

Write down the difference equations relating \( k \) in an odd period to \( k \) in the previous even period, and similarly \( k \) in an even period to \( k \) in the previous odd period. Using these two equations, analyze the dynamics of the economy. Draw a diagram showing how \( k \) will evolve from some initial condition (you may want to draw several of these, to show what can happen in different cases). Also solve algebraically for the “steady state” values of \( k \) in odd and even periods.

30) (Core exam: 1996) Consider the following small open economy OLG model.

The world interest rate is >0. There is no depreciation or population growth.

A country is perfectly open to the world capital market. The production function in the country is

\[ Y_t = K_t^5 (e_t L_t)^5 \]

Where \( Y_t \) is output at time \( t \), \( K_t \) is capital stock, \( L_t \) is labor supply, and \( e_t \) is a measure of efficiency units per worker.

\( e \) increases at rate \( g \): \( e_{t+1} = (1+g) e_t \quad g>0 \)
Every period, a new generation is born. People live two periods: in the first they work and consume. In the second they do not work and they consume their savings. There are no bequests or intergenerational transfers. People have lifetime utility functions:

\[ V = \ln(C_1) + \ln(C_2) \]

where \( C_1 \) is consumption in the first period of life and \( C_2 \) is consumption in the second period of life.

Derive an expression showing whether, in steady state, this country will have positive, negative, or zero net foreign assets.

31) (Final 2001) Consider the following OLG economy. People live for two periods. They work only when they are young, and consume in both periods. The economy is closed. The utility function is

\[ U = \ln(c_1) + \ln(c_2). \]

There is no population growth. The production function (in per worker terms) is

\[ y = A k^\alpha \]

where \( \alpha = \frac{1}{2} \). \( A \) is not constant. Rather, \( A \) takes the value \( A_o \) in odd periods and \( A_e \) in even periods, where

\[ A_o > A_e \]

The depreciation rate is 100%. That is, capital fully depreciates after use.

A) [15 points] Solve for steady state level of output in even period and in odd periods. In which periods is output higher?

B) [10 points] Who has higher utility: workers born in odd periods or workers born in even periods?

32) (old core exam question) Consider the following variation on the Solow model. There is no technological change or population growth. Output is produced using capital and labor. Capital depreciates at rate \( \delta \). Production is Cobb-Douglas. There is, however, a productive externality that takes effect when the level of capital per worker is above some level \( k^* \). Thus the production function is:

\[ y = \begin{cases} k^\alpha & \text{if } k < k^* \\ Ak^\alpha & \text{if } k \geq k^* \end{cases} \]
where A>1 and 0<\alpha<1

Suppose that the world is composed of many countries, but that all of these countries have closed economies. Assume that countries have one of three saving rates, s_l (low), s_m (middle), or s_h (high). The following is true about these saving rates:

s_l < s_m < s_h

s_l A k^{\alpha} < \delta k^*

s_m k^{\alpha} < \delta k^* < s_m A k^{\alpha}

\delta k^* < s_h k^{\alpha}

Each country also has initial capital stock per worker k_0. Assume that there is wide variation in initial capital stocks, and further that there is no relation between a country's saving rate and its initial capital stock.

Analyze how the time paths of the capital stocks of the different countries will depend on their saving rates and initial capital stocks. Pay particular attention to the questions of "convergence" (that is, to what extent will countries converge to the same steady state) and multiple equilibria. Under what conditions will countries with the same saving rates end up at the same steady state? When will countries with the same initial capital stock end up at the same steady state?

32.5) Consider an economy described by the Solow model. The production function is

Y = AK^{1/3} L^{2/3}

A grows exogenously at a rate of 2% per year. L grows exogenously at a rate of 1% per year. The economy is closed, and the fraction of output saved (and invested) is 20%. The depreciation rate is 1% per year.

At time zero, the level of output is 40 and the capital stock is 100.

At time zero, an economist conducts a "growth accounting" exercise, by starting with the production function, taking logs, and differentiating with respect to time.

$$\frac{\dot{Y}}{Y} = \frac{\dot{A}}{A} + \frac{1}{3} \frac{\dot{K}}{K} + \frac{2}{3} \frac{\dot{L}}{L} = 0$$

The economist interprets the first term on the right hand side of this equation as "growth
due to technology," the second as “growth due to capital accumulation,” and the third as “growth due to population increase.”

A. What fraction of total growth at time zero is due to capital accumulation, using this method?

B. Suppose that instead of the calculation in part A, the researcher reasoned as follows: In this model, growth of output can be due to three things:
   (i) technological progress
   (ii) population growth
   (iii) accumulation of capital beyond what is due to technological progress and population growth (that is, accumulation of capital due to movement of the economy toward its steady state).

   The economist figures out the part of capital growth due to cause (i) and decides to count growth in output resulting from this sort of capital growth as part of “growth due to technological progress” rather than “growth due to capital accumulation.” Similarly, she decides to count output growth that results from capital growth due to cause (ii) as part of “growth due to population increase” rather than “growth due to capital accumulation.” She counts only growth in output due to growth in capital of type (iii) as part of “growth due to capital accumulation.”

   Using this approach, what fraction of growth at time zero is due to capital accumulation?

33) (old midterm question) Consider the following variation on the Solow model. Suppose that the true production function (in per capita terms) is:

\[ y = A_1 k^n + A_2 k \]

There is no exogenous technological change. Population grows at rate \( n \) and capital depreciates at rate \( \delta \).

Assume that all countries in the world have the same values for \( A_1 \) and \( A_2 \), and that they all have the same saving rate. Countries differ only in their initial level of capital per person.

Discuss the extent to which countries with different initial levels of the capital stock will or will not converge over time. Distinguish, if appropriate, between different special cases based on the values of the parameters.

34) (old core exam question) Consider an augmented Solow-style model which includes human capital. Let \( k \) be the level of physical capital per worker, \( h \) be the level of human capital per worker, and \( y \) be the level of output per worker. Both human and physical capital depreciate at rate \( \delta \). There is no population growth. Let \( s_k \) be the fraction of output invested in physical capital, and \( s_h \) be the fraction of output invested in human capital. The production function is

\[ y = A_k k + A_h h \]
where \( A_k \) and \( A_h \) are two constants. Assume that \( s_k A_k < \delta \) and that \( s_h A_h < \delta \).

Analyze the dynamics of this model using a phase diagram in \( h-k \) space. Describe the paths of \( h \) and \( k \) from different initial positions. Under what condition does the model produce "endogenous growth," and what happens when this condition is not met?

35) Consider the following model. Output is produced according to the production function \( y = A_k \), where \( y \) is output per capita and \( k \) is capital per capita. There is no technological progress. Population grows at rate \( n \). A constant fraction, \( s \), of output is saved.

Capital depreciates in an unusual fashion: every period, \( d \) units of capital per person depreciate.

Analyze the dynamics of this economy. Describe the conditions under which there will be steady states, endogenous growth, etc. Calculate the long-run rate of growth, if there is one.

36) A country has production function \( y = A_k \), where \( y \) is output per person and \( k \) is capital per person. Capital depreciates at rate \( \delta > 0 \). There is no population growth. Consumption is chosen by a social planner who maximizes

\[
U = \int_0^\infty e^{-\theta t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt
\]

where \( c(t) \) is consumption at time \( t \), \( \theta > 0 \) is the discount rate and \( \sigma > 0 \). The capital stock at time 0 is \( k(0) \). Solve for the optimal value for initial consumption, \( c(0) \).

36.5) (final exam, 2003) Consumption in an economy is determined by a Ramsey-style social planner who has log utility and time discount rate \( \vartheta = 0.05 \). The rates of depreciation and population growth are both zero. Time is continuous. The production function is

\[ y = A_k \]

From time zero to time 1, the value of \( A \) is 0.05. At time 1, \( A \) will remain at 0.05 with probability 50% and will rise to 0.10 with probability 50%.

What can you say about the growth rate of output between time zero and time 1? You don't have to give exact initial value or time path. Just say whether output is rising, falling, or constant, and explain why. [Note: this problem is harder than it looks.]
37) Consider the following OLG model. Individuals live for two periods. There is a new generation born every period.

In the first period, individuals supply one unit of labor inelastically. They do not consume in the first period. In the second period, they do not work, and consume their first period savings along with accumulated interest. Individuals are born and die with zero assets. There is no intergenerational altruism and no population growth.

Production is carried by identical small firms. Firm i has production function:

\[ Y_i = B K_i^\alpha L_i^{1-\alpha} \]

where \( K_i \) and \( L_i \) are the quantities of labor and capital used by firm i. Firms pay factors their marginal products.

The aggregate productivity coefficient (B), is determined by the aggregate capital stock

\[ B = A K^{1-\alpha}, \]

where \( K \) (without a subscript) is the total capital stock in the economy. Firms do not take the effect of their capital stock on \( B \) into account in their production decisions.

Solve for (and describe) the steady-state of the model. If there is either a steady-state level of output or a steady state growth rate of output, solve for it. If not, explain why not.

39) Consider a variation on the Lucas model analyzed in class.

Production (in per capita terms) is

\[ y = k^{0.5} h^{0.5} \]

where \( k \) and \( h \) are per capita levels of physical and human capital.

Physical capital is accumulated according to

\[ \dot{k} = s_k y - \delta k \]

Human capital is accumulated according to
\[ \dot{h} = s_h y - \delta h \]

Further, it just so happens that \( s_k = s_h = \delta \)

A. What is the steady state growth rate of output?

B. Suppose that a country is in steady state, when suddenly 3/4 of its physical capital (but none of its human capital) is destroyed. Will the growth rate of output immediately after the shock be higher, lower, or the same as the growth rate in steady state?

40) Three countries, A, B, and C, are all described by the version of the Lucas model presented in class:

\[
\begin{align*}
y &= k^\alpha (uh)^{(1-\alpha)} \\
\dot{h}/h &= \phi (1-u) \\
\dot{k} &= sy - (n+\delta)k
\end{align*}
\]

Initially, all three countries have the same levels of \( s, u, h, \) and \( k \). In year \( t \), dictators seize power in countries A and B and engage in policy experiments. Country A raises the rate of saving to some higher level, holding \( u \) constant. Country B reduces \( u \) to some lower level, holding \( s \) constant. Country C keeps both \( s \) and \( u \) constant. After 10 years, the dictators are overthrown, and \( s \) and \( u \) are returned to their original levels in countries A and B.

Draw a graph showing the time paths of the growth rates of output in the three countries during and after the period of policy experimentation. Distinguish between different special cases based on the parameters, if appropriate. Draw a second graph showing the time paths of the log of output in the three countries.

40.5) [midterm exam, 2004] Consider a variation of the Lucas growth model. The production function is

\[ y = (u_k k)^\alpha (u_h h)^{(1-\alpha)} \]

Where \( u_k \) is the fraction of physical capital that is used in producing output and similarly \( u_h \) is the fraction of human capital that is used in producing output.

The accumulation equations for \( k \) and \( h \) are
\[ \dot{k} = \phi \ (1-u_k)k - (n+\delta)k \]
\[ \dot{h} = \phi \ (1-u_h)h - (n+\delta)h \]

A) solve for the steady state growth rate of output

B) Suppose that \( \alpha = .5, \ u_k = u_h = .5, \ (n+\delta = .05), \ \phi = 0.1 \). A country is in steady state. Now suppose that \( u_k \) falls to .125, while \( u_h \) remains constant. How many years (approximately) will it take until output regains the level it would have had if there had been no change in \( u_k \)? [Helpful hint: \( \ln(2) \approx 0.70 \).]

41) (core exam: 1996) Consider the following hybrid of the Solow model and the Lucas model with human capital.

Output is produced with physical capital, human capital, and labor according to the production function (written in per-worker terms):

\[ y = k^\alpha \ [(1-u)h]^{1-\alpha} \]

where \( y \) is output per worker, \( h \) is human capital per worker, \( k \) is physical capital per worker, and \( (1-u) \) is the fraction of their time that workers spend producing output.

There is no population growth. Physical capital accumulates according to

\[ \dot{k} = sy - \delta k \]

where the saving rate, \( s \), is exogenous and fixed. When workers are not producing output, they are producing human capital (there is no leisure). The only input required to produce human capital is time (unlike the Lucas model, where human capital itself is one of the inputs to human capital production). Human capital also depreciates at the same rate (\( \delta \)) as physical capital. Thus, the evolution of human capital per capita is given by:

\[ \dot{h} = u - \delta h \]

Assume that \( s = \delta \).

A. Describe the steady state of the model. Is it one with a constant level of output or with a
constant growth rate of output?

B. Solve for the level of \( u \) that maximizes either the level or the growth rate of output (depending on which is constant in steady state).

C. Let \( u^* \) be the level of \( u \) that you solved for in part B. Suppose that the economy is in steady state with \( u < u^* \). Suddenly, \( u \) jumps up to \( u^* \). Use a phase diagram in \((h,k)\) space to analyze the behavior of \( h \) and \( k \). Draw time series pictures showing how \( h, k \) behave during the transition to the new steady state.

42) [midterm exam 2002] Consider the problem of a social planner choosing the fraction of the labor force to be devoted to R&D and the fraction that will work. The population is constant and is normalized to one. The social planner has the standard discounted utility function

\[
V = \int_0^\infty e^{-\theta t} \ln(c(t)) \, dt
\]

There is no physical capital in the model. Consumption is equal to production, which is given by the production function

\[ y = h u \]

where \( h \) is the level of human capital and \( u \) is the fraction of their time that workers spend producing output. The growth rate of human capital is given by

\[
\frac{\dot{h}}{h} = \phi (1 - u) \]

The level of human capital at time zero is equal to one. At time zero, the social planner must choose a value of \( u \) that will be constant forever.

Finally, we assume that \( \theta < \phi \).

Your task is to solve for the optimal level of \( u \).

**Big hint:** To solve this problem, you do not need to think about first order conditions or dynamic optimization or things like that. Rather, you can just write out the present discounted value of utility as a function of the choice of \( u \) and maximize.

**Essential helpful fact:**
\[
\int_0^\infty t e^{\theta t} dt = \frac{1}{\theta^2}
\]

(To get this fact you have to do a bunch of limits and other junk – it is not something that I would expect you to know off the top of your head.)

43) [final exam 2002] This question extends the previous problem to deal with the issue of technology spillovers. In that problem, you considered the problem of a social planner choosing the fraction of the labor force to be devoted to R&D and the fraction that will work. The population is constant and is normalized to one. The social planner has the standard discounted utility function

\[
V = \int_0^\infty e^{-\theta t} \ln(c(t)) dt
\]

There is no physical capital in the model. Consumption is equal to production, which is given by the production function

\[y = h u\]

where \(h\) is the level of human capital and \(u\) is the fraction of their time that workers spend producing output. The growth rate of human capital is given by

\[
\frac{\dot{h}}{h} = \phi(1-u)
\]

The level of human capital at time zero is equal to one. At time zero, the social planner must choose a value of \(u\) that will be constant forever.

Finally, we assumed that \(\theta < \phi\).

Your task was to solve for the optimal level of \(u\).

Now consider a model where there are technological spillovers from a leader country to a follower country. Unlike the version of the model that we considered in class, the effect of the spillover does not depend on how large the gap in technology is – a follower benefits as long as it is even slightly behind the leader. Specifically, let the growth rate of technology in the leader country be

\[
\left(\frac{\dot{h}}{h}\right)_L = \phi_L(1-u_L)0
\]
while in the follower it is

\[
\begin{pmatrix}
\dot{h}_F \\
\dot{h}_F
\end{pmatrix}
= \phi_F (1 - u_F)0
\]

where \( \theta < \varphi_L < \varphi_F \)

There are two countries, labeled 1 and 2. At time zero, the two countries have the same level of \( h_0 \), which for convenience we set equal to one.

In country 1, the value of \( u \) is fixed at some level \( u_1 \), and will not be affected by the action of country 2. The social planner in country 2 has the same infinite horizon utility function we introduced above. She is able to choose some level of \( u_2 \). **However**, the social planner in country 2 can only choose a single value of \( u_2 \) that will hold for all points in time starting at time zero. She cannot choose a path where \( u_2 \) changes over time.

Finally, assume that the level of \( u \) in country 1 is set at the level that the social planner in country 2 would have chosen if there were no technology spillovers between countries.

A. [10 points] Draw a diagram showing the growth rate of output in country 2 as a function of \( u_2 \). Calculate the value of \( u_2 \) at which there is a the key inflection point.

B. [10 points] Draw a diagram showing the present discounted value of utility in country 2 as a function of the choice of \( u_2 \). Label the optimal choice.

44) Consider an economy in which there are two kinds of capital: private capital, consisting of machines, buildings, etc., and government capital, consisting of highways, dams, etc. The production function is:

\[
y = k^{0.5} z^{0.5}
\]

where \( y \) is the per capita level of output, \( k \) is the per capita level of private capital, and \( z \) is the per capita level of government capital. Both private capital and government capital depreciate at rate \( \delta \). There is no population growth or technological change.

The government collects a constant fraction, \( \tau \), of output, which it spends on building government capital. Of the remaining output, a constant fraction, \( s \), is invested in building private capital.

Show that \( k/z \) converges to a steady state along the balanced growth path. Find the growth rate of the economy when \( k/z \) is at this steady state level. Find the level of taxes, \( \tau \), which maximizes the steady-state growth rate of output.
45) (old core exam question) Consider a country with production function:

\[ Y = K^{0.5} H^{0.5} \]

where \( K \) is the stock of physical capital and \( H \) is the stock of human capital. There is no depreciation, population growth, or technological change. Physical capital is accumulated according to:

\[ K = s_k Y \]

and human capital is accumulated according to:

\[ H = s_h Y \]

A country is growing at a constant, positive rate along its balanced growth path. At a point in time, the country's saving rate in physical capital (\( s_k \)) doubles, while it's saving rate in human capital (\( s_h \)) remains constant.

[Note: you do not have to answer part A in order to answer part B]

A. What will be the effect of this change on the growth rate of output along the new balanced growth path? That is, by what factor will the growth rate of output along the new balanced growth path differ from that along the initial balanced growth path?

B. What will be the instantaneous effect of this change on the growth rate of output? That is, by what factor will the growth rate of output change instantaneously?

46) (Midterm exam, 2001) Consider the Aghion and Howitt model of technology creation. Suppose that there are only a fixed number of possible new inventions. Call this number \( T \). Once the \( T \)th new technology has been invented, there will never be another new technology to replace it. Further, suppose that patents last forever, so that the firm which invents the \( T \)th new technology will have a monopoly on producing the intermediate good forever. Obviously, once the \( T \)th new technology has been invented, the labor force devoted to R&D will be zero. Now, consider how much labor will be devoted to R&D during the epoch of the \( T-1 \) technology, the \( T-2 \) technology, etc. To be more concrete, let us compare R&D labor force in these periods to R&D labor force that would be observed in the steady state if there were no limit to the number of new technologies. How would the R&D labor force in epoch \( T-1 \) compare to the steady state of the regular model? How about the R&D labor force in epoch \( T-2 \)? \( T-3 \)? \( T-4 \)? Etc.

I do not want you to go through the whole model to answer this question. Based on the logic of the model it should not be too hard to see the answer.
47) Consider the following model with endogenous technology and population.

Population growth depends on the difference between current income per capita, $y$, and (which is some sort of subsistence level):

$$ n = \beta(y - \bar{y}) $$

Output is produced with labor ($L$) and land ($R$).

$$ Y = (AR)^{\alpha} L^{1-\alpha} $$

Technology growth depends on output per capita:

$$ \dot{A}/A = \gamma y \quad 0 < \gamma < \beta $$

Analyze the model's dynamics and steady state(s). Solve for any steady state values of output per capita.

48) A researcher examines cross country data. She fits the following equation for growth rates:

$$ = .01 - .02 \ln(y) + .03 \ln(s) + .04 \text{DEM} $$

Where is the growth rate of income per capita, $s$ is the saving rate, and DEM is a dummy variable that takes the value 1 if the country is a democracy and zero if it is not.

Assume that the world is properly described by the Solow model (with no human capital), but that in addition, the form of government affects the production function.

Calculate the steady state difference in income per capita between a democracy and a non-democracy which have the same saving rate.

48.5) [midterm exam, 2004] Consider the partial equilibrium problem of an individual deciding how many children to have. He is endowed with one unit of time, which he can spend working or raising children. His wage per unit time, $w$, is exogenous. The cost of raising children is $z$ units of time per child [he is able to have non-integer numbers of children]. His utility function is
\[ V = \ln(c - c^*) + \ln(n) \]

Where \( c^* \) is the subsistence level of consumption (we assume that \( w > c^* \)).

Trace out and describe as fully as possible the relationship between the wage and fertility.

49) (midterm exam, 2001) In our analysis of the Lucas model of population, we assumed that the "price" of children was in terms of goods. Suppose instead that the price of children is in terms of time. Specifically, suppose that raising one child takes \( z \) units of time.

An economy is composed of individuals who are adults for one period. In that period they work, consume, and raise children. (Children do not do anything in the model other than use up time from their parents.) They are endowed with one unit of time, which is used for either working or child-raising (there is no leisure). The number of children per person is \( n \). The budget constraint is thus

\[ c = w(1-zn) \]

where \( w \) is the wage.

People get utility from consumption and the number of children:

\[ U = \ln(c) + \ln(n) \]

However, there is also a minimum level of consumption, \( c^* \), below which \( c \) cannot fall (think of this as being the level of consumption necessary for subsistence.) That is, \( c \geq c^* \).

A) Solve for the optimal levels of fertility and consumption as functions of the wage. Draw a diagram with \( n \) on the vertical axis and \( c \) on the horizontal axis, and show how changing the wage affects the budget constraint and the optimal choices of fertility and consumption. If there are any critical levels of wage where the behavior changes, solve for them.

B) Now, suppose that we embed this model of population in a very simple growth model. Production uses land and labor. The quantity of land in the economy, \( X \), is fixed. The production function is

\[ Y = (L(1-zn))^{1/2} X^{1/2} \]

where \( L \) is the number of adults. We assume that land earns no return, so workers are paid their average product.
Assume that $z < \frac{1}{2}$.

What are the steady state levels of consumption and fertility? Explain how you know this. You should be able to figure this out very easily, without doing a lot of math.

50) [midterm exam, 2005] Consider a country described by the following model. The production function is

$$y = k^\alpha (uh)^{1-\alpha} x^{1-\alpha}$$

Where $k$ is capital per worker, $h$ is human capital per worker, $u$ is the fraction of their time that workers spend producing output, and $x$ is land per worker. Population grows at rate $n$ and capital depreciates at rate $\delta$.

Capital accumulation is given by the standard equation

$$\dot{k} = sy - (n + \delta)k$$

Human capital accumulates according to the equation

$$\dot{h} = \phi h (1-u) - nh$$

(Notice that I have added a term for “dilution” of human capital per worker due to population growth.)

The fraction of time spent working ($u$) and the saving rate ($s$) are taken as exogenous.

Analyze this model in a diagram with $\frac{k}{xh}$ on the horizontal axis and the three measures $\frac{\dot{k}}{k}$, $\frac{\dot{h}}{h}$, and $\frac{\dot{x}}{x}$ all measured on the vertical axis. Indicate the steady state and briefly describe its stability properties.