Several ways to approach this subject.

1. Note that “saving” and “consumption” are really the same question: that is, you get a certain amount of income, and you can save it or consume it. So can’t think about one without thinking about the other.

2. This topic is really part of both the long run and the short run analysis. In the long run, we will see later this semester, the saving rate determines the level of output (or the growth rate or output). But in the short run, as you will see in the second semester, the determination of consumption is also important for studying the business cycle.

2.5 Consumption theory is one of the most elegant branches of economic theory. Much of the approach taken here to consumption is taken elsewhere in economics to e.g. fertility, schooling, health, etc. Thus these tools (and the problems with them) are far more general than it might appear.

3. In all of this section of the course, we will be treating labor income as exogenous (Note: “exogenous” does not mean “constant” or “certain.”) We will also mostly treat interest rates as being exogenous, but also look at some cases of endogenous interest rates.

You may recall the approach taken to consumption in many undergraduate macro textbooks is to think about a “consumption function" that relates consumption to disposable income:

\[ C = C(Y-T) \]  
[where note that we are using c as both the name of the function and the name of the thing it is determining.]

often this is written in a linear form:

\[ C = c_0 + c_1(Y-T) \]

where the little c's are coefficients. \(c_1\) is, of course, the marginal propensity to consume. [picture]

This is often called the “Keynesian” consumption function. Keynes wrote that \(c_0>0\) and \(0<\text{c}_1<1\) due to a “psychological law” -- essentially that when you do not have a lot of income, you focus on immediate needs; but when you have satisfied these, you look more to the future and save.
Ways to test: look cross sectionally; look at short time series. (flesh his out)
Both of these looked good for the Keynesian consumption function.

Two problems with the Keynesian view:

1. Empirical: What does this model predict will happen to the rate of saving as a country gets richer?

\[ s = \frac{Y - T - C}{Y - T} = I \cdot \frac{1 - c_1}{Y - T} = I \cdot \frac{c_0}{Y - T} - c_1 \]

where \( C/(Y-T) \) is often called the average propensity to consume. So the Keynesian consumption function says that as a country gets richer the saving rate should rise. This just doesn't work. The saving rate is pretty constant over long periods of time.\(^{12}\)

2. Theory: Think about the act of saving: you are moving consumption from one period to another. Thus saving should be viewed explicitly as an intertemporal problem. So for example, the MPC should depend on why your income has gone up. Put another way, the consumption function should have a lot more than just today's income in it -- for example, it should have tomorrow's income in it.

So we want a model of saving behavior that is more based on fundamentals. To build a such a model, we start with the question: Why do people consume? Answer: Because it makes them happy.

We represent the idea that consuming makes people happy with a utility function.

By utility function, we just mean some function that converts a level of consumption into a level of utility. \( U = U(C) \) [picture]

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1 Historical note: This wasn't actually known for sure when Keynes wrote: Simon Kuznets, who invented national income accounting -- ie how to measure GDP and stuff -- discovered the approximate constancy of the US saving rate over a period of 100 or so years. His discovery set off a flurry of work on consumption in the 1950s that culminated in Friedman and Modigliani’s contributions. Interestingly, in most other developed countries (summarized in Angus Maddison's work) the saving rate has risen over time -- although probably not in the way that Keynes' model predicted.

2 The Kuznets finding can be put another way. If we go to the data (say annual data on income and consumption for a country over time) and run the regression \( C = c_0 + c_1 \ Y \), we will get the result that in short samples the estimated value of \( c_1 \) will be smaller than it will be in large samples. When we talk about the Permanent Income Hypothesis we will see why this is true.
Why should the utility function be curved? Try to motivate intuitively: think about the marginal utility of additional consumption. Seems like this goes down.

Most of the interesting things that we can say about utility come from thinking about two issues: how we add up utility over many different periods of time, and how we deal with the expected utility when there is uncertainty.

**Adding up Utility Over Time**

How do we add up utility across time? Well essentially, we can just take the sum of individual utilities. Say that we are considering just two periods. Let $U( )$ be the “instantaneous” utility function. Then total utility, $V$, is just

$$V = U(C_1) + U(C_2)$$

(In a little while we will introduce the notion of *discounting*, by which utility in the future may mean less to us than utility today. But for now, we will ignore this idea.)

What does our understanding about the utility function say about the optimal relation between consumption at different periods of time. Say, for example, that we have $300 to consume over two periods (and we temporarily ignore things like the interest rate): How shall we divide it up?

The answer is that we would want to *smooth* it – that is, consume the same amount in each period. The way to see this is to look at the marginal utility of consumption. Suppose that we consumed different amounts in different periods. Then the marginal utility of consumption would be lower in the period where we consumed more. So we could consume one unit less in that period, and one unit more in the period where the marginal utility was higher, and our total utility would be higher.

**Utility Under Uncertainty**

Now let's consider a case where there is only one time period, but in which there is uncertainty about what consumption will be in that period. Suppose, for example, that I know that there is a 50% chance that my consumption will be $100 and a 50% chance that my consumption will be $200. How do we calculate my expected utility?

There are two ways that you might consider doing it: could take the expected value of my utilities, or the utility of my expected consumption.

$$V = .5U(100) + .5U(200)$$
or \( V = U(.5*100 + .5*200) \)

The first of these methods of calculating utility from an probabilistic situation is called Von Neumann - Mortgenstern (VNM) utility. This is the approach that we always use. The second method is called wrong.

How do we know the VNM utility is the right way to think about utility when there are different possible states of the world? Here is a simple demonstration: Suppose that you can have either $150 with certainty, or a lottery where you have a chance of getting either $100 or $200, each with a probability of .5. Which would you prefer? Almost everyone would say they prefer the certain allocation. This is a simple example of risk aversion. But notice that if we chose the second technique for adding up utility across states of the world, we would say that you should be indifferent.

The fact that uncertainty lowers your utility is called risk aversion. Notice that risk aversion is a direct implication of the utility function being curved. (The mathematical rule that shows this is called Jensen's inequality: if \( U \) is concave, then \( U(E(C)) > E(U(C)) \), where \( E \) is the expectation operator.) If the utility function were a straight line then the utility of $150 with certainty would be the same as the utility of a lottery with equal chances of getting $100 and $200. A person who indeed gets equal utility from these two situations is called risk neutral.

What are the consequences of risk aversion? Clearly this is the motivation for things like insurance, etc. Similarly, this is why in financial theory we say that people trade off risk and return: to accept more risk, an investor has to be promised a higher expected return.

**The Relation Between Risk Aversion and Consumption Smoothing**

Now we get to the really big idea: risk aversion and consumption smoothing are really two sides of the same coin: they are both results of the curvature of the utility function. If the utility function were linear (and so the marginal utility of consumption constant) then people would not care about smoothing consumption, and their expected utility would not be lowered by risk.

This will be important for many reasons: among them is that even when we are talking about a world with no uncertainty, we will often use the idea of risk aversion to measure the curvature of the utility function.

**The CRRA Utility Function**

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3One can come up with many instances of risk neutrality or even risk-loving (i.e. more uncertainty raises utility) behavior, such as participating in lotteries, flipping a coin with your friend for who will buy coffee, etc. However, it is unlikely that these exceptions tell us much about the vast majority of consumption decisions.
We will often use a particular form of the utility function, called the **Constant Relative Risk Aversion** utility function.

\[ U(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma} \]

where \( \sigma > 0 \). Note that if \( \sigma > 1 \), then the CRRA formulation implies that utility is always negative, although it becomes less negative as consumption rises. This does not matter, although it often gets students confused.

Note that in the special case where \( \sigma = 1 \), the CRRA utility function collapses to \( U(C) = \ln(C) \).\(^4\)

\( \sigma \) is called the **coefficient of relative risk aversion** and it measures, roughly, the curvature of the utility function. If \( \sigma \) is big, then a person is said to be risk averse. If \( \sigma \) is zero, the person is said to be risk-neutral.

To see how \( \sigma \) measures the curvature of the utility function, we can calculate the elasticity of marginal utility with respect to consumption, that is

\[^4\text{Proof: First re-write the utility function by adding a constant: } u(c) = (c^{1-\sigma}-1)/(1-\sigma). \text{ Think of this as a function of } \sigma: g(\sigma) = (c^{1-\sigma}-1)/(1-\sigma) \text{ re-write as} \]

\[ g(\sigma) = \frac{ce^{\ln(c)\sigma} - 1}{1-\sigma} \]

since \( g(1) = 0/0 \), we apply L'Hopital’s rule

\[ \lim_{\sigma \to 1} g(\sigma) = -c \frac{\ln(c)e^{\ln(c)\sigma}}{\ln(c)} = \ln(c) . \]
So the bigger is $\sigma$ (in absolute value), the more rapidly the marginal utility of consumption declines as consumption rises [picture]. And the larger is this change in marginal utility, the greater is the motivation for consumption smoothing, insurance, etc.

As an exercise, we can show this by calculating the amount that a person is willing to pay to avoid uncertainty. For example, calculate the value $x$ such that the utility of $150-x$ with certainty is equal to the utility of a 50% chance of $100 and a 50% chance at $200. How does $x$ change with $\sigma$?

We solve:

$$
(150 - x)^{1-\sigma} = .5 \times 100^{1-\sigma} + .5 \times 200^{1-\sigma}
$$

$$
x = 150 - (.5 \times 100^{1-\sigma} + .5 \times 200^{1-\sigma})^{\frac{1}{1-\sigma}}
$$

We can use a calculator to find the value of $x$ for different values of $\sigma$. By thinking about what value of $x$ seems reasonable, we can then decide what is a reasonable value for $\sigma$ (see table). For example, if $\sigma = 6$, then $x=35.8$, so a person would be indifferent between a 50% chance of consuming $100 and a 50% chance of consuming $200, on the one hand, and certain consumption of $114.20, on the other.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8.6 (log utility)</td>
</tr>
<tr>
<td>2</td>
<td>16.7</td>
</tr>
<tr>
<td>3</td>
<td>23.5</td>
</tr>
<tr>
<td>4</td>
<td>28.8</td>
</tr>
<tr>
<td>5</td>
<td>32.9</td>
</tr>
<tr>
<td>6</td>
<td>35.8</td>
</tr>
</tbody>
</table>

Note that even though log utility is probably not reasonable a priori (based on the above) or empirically, we use it a lot because it is so convenient. Empirical estimates of $\sigma$ probably average around 3, but there is no agreement. Some anomalies in finance (such as the "equity premium puzzle") can only be explained with what seem like unreasonably high values of $\sigma$. 

\[
\frac{dU'}{dc} = \frac{U''}{U} = -\sigma
\]
[see homework question on CRRA and CARA.]

[ The difference between “relative risk aversion” and “absolute risk aversion” can be thought of this way. Suppose that I am willing to pay 10 to avoid the uncertainty of a lottery that gives me either 150 or 50 each with probability 50% – that is, I consider certain consumption of 90 to have utility equal to the lottery. If utility is CRRA, then I will also be willing to pay 100 to avoid a lottery of 1500 or 500. If utility is CARA, then I will be willing to pay 10 to avoid a lottery of 1050 or 950. Note that with CARA I will be willing to more than 100 to avoid a lottery of 1500 or 500 – this should be obvious for the following reason: the larger is uncertainty, the more (at the margin) you are willing to pay to avoid it. So if a CARA consumer with expected income of 1000 will pay 10 to avoid 50 worth of uncertainty, he will pay more than 100 to avoid 500 worth of uncertainty.]

**Fisher Model**

So now we look more formally at an intertemporal model of saving. The simplest model is the two-period model of Irving Fisher.

People live for two periods. They come into the world with no assets. And when they die, they leave nothing behind.

In each period they have some wage income that they earn: $W_1$ and $W_2$.

Similarly, in each period, they consume some amount $C_1$ and $C_2$.

The amount that they save in period 1 is $S_1 = W_1 - C_1$. $S_1$ can be negative, in which case they are borrowing in the first period and repaying their loans in the second period.

For the time being assume that they do not earn any interest on their savings or pay any interest on their borrowing. So the amount that they consume in the second period is

$$C_2 = S_1 + W_2$$

that is, in the second period you consume your wage plus your savings.

We can combine these two equations to get the consumer's intertemporal budget constraint:

$$C_1 + C_2 = W_1 + W_2$$

We can draw a picture with $C_1$ on the horizontal axis and $C_2$ on the vertical axis. The budget constraint is a line with a slope of negative one. Note that the budget constraint runs through the point $W_1$, $W_2$ -- you always have the option of just consuming your income in each period. The Y and X intercepts of the budget constraints are both $W_1 + W_2$. 

7
Consumer can consume any point along this line (or any beneath it, but that would be waste).

What is saving in this picture? Show which points involve saving or borrowing.

So where does the person choose to consume? Well, clearly along with a budget constraint we are going to need some indifference curves.

Say that his total utility (V) is just the sum of consumption in each period:

\[ V = U(C_1) + U(C_2) \]

Where \( U() \) is just a standard utility function.

To trace out an indifference curve, consider a point where \( C_1 \) is low and \( C_2 \) is high. At such a point, the marginal utility of first period consumption is high, and that of second period consumption is low. So it would take only a small gain in \( C_1 \) to make up for a big loss of \( C_2 \) in order to keep the person having the same utility. So the indifference curve is steep. Similarly, when \( C_1 \) is large and \( C_2 \) is small, the indifference curve is flat.\(^5\) So it has the usual bowed-in shape.

So optimal consumption is where the budget constraint is tangent to an indifference curve.

We can also solve the problem more formally, setting up the lagrangian:

\[ L = U(C_1) + U(C_2) + \lambda(W_1 + W_2 - C_1 - C_2) \]

and taking the first order conditions:

\[ \frac{dL}{dC_1} = 0 = U'(C_1) - \lambda \quad \Rightarrow \quad \lambda = U'(C_1) \]

\[ \frac{dL}{dC_2} = 0 = U'(C_2) - \lambda \quad \Rightarrow \quad \lambda = U'(C_2) \]

so \( C_1 = C_2 \)

Combining this with the budget constraint gives: \( C_1 = C_2 = \frac{(W_1 + W_2)}{2} \), which is not so shocking, when you think about it.

\(^5\) More formally, one can use the implicit function theorem. Let \( F(C_1, C_2) = U(C_1) + U(C_2) \). Then for \( F(C_1, C_2) = k \) (where \( k \) is some constant):

\[ \frac{d C_2}{d C_1} = \frac{F_{C_2}}{F_{C_1}} = \frac{U'(C_1)}{U'(C_2)} \]
We can use this simple model to think about consumption in the face of different circumstances.

What happens if income rises? This will shift out budget constraint. Consumption in both periods will rise. What happens to saving? Answer: it depends on which periods income went up.

-- Suppose that your current income falls but that your future income rises by exactly the same amount. How should consumption change? How about saving?

Already, we can see some problems with Keynes’ way of looking at consumption. Consumption depends not just on today’s income but on future (or past) income.

**Interest rates**

Now we make the model slightly more complicated by considering interest rates:

let $r$ be the real interest rate earned on money saved in period 1 -- or the interest rate paid by people who borrow in period one.

Now the definition of saving is still:

$S_1 = W_1 - C_1$

but consumption in the second period is now:

$C_2 = (1+r) S_1 + W_2$

or combining these:

$W_1 + W_2/(1+r) = C_1 + C_2/(1+r)$

Can draw diagram as before

Y intercept is $(1+r)W_1 + W_2$.
X intercept is $W_1 + W_2/(1+r)$.

The budget constraint still goes through the point $(W_1, W_2)$, which we call your “endowment point” - - that is, if you consume your income in each period, that is a feasible consumption plan no matter
what the interest rate is.

What happens to the budget constraint when the interest rate changes??

Answer: it rotates around the endowment point. What does this do to saving in the first period (ie to consumption in the first period?)

Answer is: it depends.

First, lets look at what happens in the case where the person was initially saving. Remember from micro that there are two effects: the income and the substitution effect.

Income effect is that we can get onto a higher indifference curve. This tends to raise C for both periods.

Substitution effect: consumption in the second period has gotten cheaper. This tends to lower first period consumption and raise second period consumption.

Upshot is that in this case, can't tell what happens to first period consumption, or first period saving.

What if person had had negative saving in the first period, and then interest rate goes up?

Now income and substitution effects work in the same direction, so that first period consumption will fall, and saving will rise.

Discounting

We might want to introduce some discounting of utility experienced in the future. For example, suppose that $\Theta$ is some discount factor that we use for discounting future utilities.

$V = U(C_1) + U(C_2)/(1+\Theta)$

now we can once again solve for the optimal path of consumption with both interest and discounting. We set up the lagrangian:

$L = U(C_1) + \frac{U(C_2)}{1+\Theta} + \lambda \left( W_1 + \frac{W_2}{1+r} - C_1 - \frac{C_2}{(1+r)} \right)$

and get the first order conditions
\[ \frac{dL}{dC_1} = U'(C_1) - \lambda \]
\[ \frac{dL}{dC_2} = U'(C_2)/(1+\Theta) - \lambda/(1+r) \]

which can be solved for:
\[ \frac{U'(C_1)}{U'(C_2)} = (1+r)/(1+\Theta) \]

this is one equation in the two unknowns of \( C_1 \) and \( C_2 \). It can be combined with the budget constraint to give two equations in two unknowns, and so can be solved for the two values of \( C \). To do this, however, one needs to know the exact form of the utility function. This is done in one of the homework exercises.

**Liquidity Constraints**

What happens if there are constraints on borrowing? What does the budget constraint look like now?

For person who would have wanted to save anyway, no big deal. But for person who would have wanted to borrow, they will be at corner. We say that such a person is "liquidity constrained." Example of a college student.

What will such a person's consumption be? Just their current income. So they will look a lot more like the Keynesian model, except that the MPC will be one.

**Differential interest rates**

It may also be the case that the interest rate for borrowing is different than the interest rate for saving -- presumably the rate for borrowing will be higher.

What will the budget constraint look like in this case? It will be kinked at the endowment point. In this case, there are three possible optima: either tangent to one of the arms, or at the kink point. Interesting result is that if the optimum is at the kink point, then small changes in one or both interest rates will not affect the optimal level of consumption.

**Extension to More than Two Periods**

Now we can easily extend the model to an arbitrary number of periods:

Consider a person planning consumption over periods 0...T-1. (labeling the periods this way is just slightly more convenient) She faces a path of wages \( \{W_0, \ldots, W_{T-1}\} \)
she gets utility according to an instantaneous utility function \( U(C) \), which is discounted at rate \( \theta \). That is

\[
V = \sum_{t=0}^{T-1} \frac{U(C_t)}{(1 + \theta)^t}
\]

She faces interest rate \( r \) on any assets (negative or positive) that she has. In particular, call \( A_t \) the assets that she has at the beginning of a period. This is equal to

\[
A_t = (1+r)(A_{t-1} + W_{t-1} - C_{t-1})
\]

She starts life with zero assets: \( A_0 = 0 \).

We also impose the rule that she must have zero assets at the end of her life -- that is \( A_T = 0 \) (where \( A_T = (1+r)(A_{T-1} + W_{T-1} - C_{T-1}) \)). Put another way, in the last period of life she spends her earnings plus any accumulated assets (or less any accumulated debts). Dying in debt is not allowed.

How will we derive her inter-temporal budget constraint?

Start by writing down the expression for assets in each period

\[
A_1 = (1+r)(W_0 - C_0) \quad [\text{since } A_0 = 0 ]
\]

\[
A_2 = (1+r)(A_1 + W_1 - C_1) = (1+r)(W_1-C_1) + (1+r)(W_0 - C_0)
\]

etc...

\[
A_T = (1+r)(W_{T-1} - C_{T-1}) + (1+r)^2(W_{T-2} - C_{T-2}) + ... (1+r)^T(W_0 - C_0)
\]

We divide all the terms in this last expression by \((1+r)^T\), and note that it is equal to zero, to get

\[
\theta = \sum_{t=0}^{T-1} \frac{W_t - C_t}{(1 + r)^t}
\]

Notice that we have gotten rid of all of the \( A \)'s. This expression can be re-arranged to say that the present discounted value of consumption is equal to the present discounted value of wages.

\[
\sum_{t=0}^{T-1} \frac{W_t}{(1 + r)^t} = \sum_{t=0}^{T-1} \frac{C_t}{(1 + r)^t}
\]
This is the intertemporal budget constraint... which looks a lot like the two period version derived above.

**An Aside: The Budget Constraint in Continuous Time**

We can also derive a similar intertemporal budget constraint in continuous time. The evolution of assets is governed by the differential equation:

\[
\frac{d}{dt} A(t) = rA(t) + w(t) - c(t)
\]

This can be solved, along with the initial condition \(A(0)=0\), to give

\[
A(t) = \int_0^t (w(s) - c(s)) e^{r(t-s)} \, ds \tag{1}
\]

This just says that assets at time \(t\) are the present values of the past differences between wages and consumption.

Assets at the end of life are zero, that is, \(A(T)=0\). So setting \(t=T\) in the above equation,

\[
\int_0^T e^{r(T-s)} w(s) \, ds = \int_0^T e^{r(T-s)} c(s) \, ds \tag{2}
\]

Multiplying by \(e^{-rT}\)

\[
\int_0^T e^{-rs} w(s) \, ds = \int_0^T e^{-rs} c(s) \, ds \tag{3}
\]

[End of Aside]

Now with our budget constraint and our utility function, we can do a big Lagrangian....

\[
L = \sum_{t=0}^{T-1} \frac{U(C_t)}{(1+\theta)^t} + \lambda \left( \sum_{t=0}^{T-1} \frac{W_t \cdot C_t}{(1+r)^t} \right)
\]

to solve this we would just find the \(T\) first order conditions which, combined with the budget constraint, would allow us to solve for the \(T+1\) unknowns: \(\lambda\) and the \(T\) values of consumption. In many cases this is a big mess to solve, but we can get far by just looking at the FOCs for consumption in two adjacent periods, \(t\) and \(t+1\):
\[
\frac{dL}{dC_t} = \frac{U'(C_t)}{(1+\theta)^t} - \lambda \frac{1}{(1+r)^t} = 0
\]

\[
\frac{dL}{dC_{t+1}} = \frac{U'(C_{t+1})}{(1+\theta)^{t+1}} - \lambda \frac{1}{(1+r)^{t+1}} = 0
\]

these two can be combined to give

\[
\frac{U'(C_t)}{U'(C_{t+1})} = \frac{1+r}{1+\theta}
\]

this is a key condition that relates consumption in adjacent periods. Notice that even if we don't know the full solution to the consumer's problem (that is, what the level of consumption in each period should be), we know that this condition should hold.

There is a huge amount of intuition built into this expression, so it is worth thinking about for a while.

Let's start on the intuition by showing how we could have gotten a similar result without calculus:

Suppose that I have a discounted utility function, and that the interest rate is zero. I have some set amount of total consumption that I want to do. How will I divide it between the periods?

To see the answer: consider a path of consumption \((C_0, C_1, \ldots)\) Suppose that I want to know whether this path of consumption is optimal. Well, suppose that I consider consuming slightly less (call it one unit, for convenience) in period zero, and then consuming the same amount more in period one.

How much would I lose? answer: \(U'(C_0)\)

How much would I gain? answer: \(U'(C_1)/(1+\theta)\)

Note that the \((1+\theta)\) comes from the fact that utility that I get in the second period is not worth as much to me as utility in the first period. Now if one of these was bigger than the other, then clearly the path of consumption that I was considering was not the optimal one. So everywhere along the
optimal consumption path, it will be the case that
\[ U'(C_t) = U'(C_{t+1})/(1+\Theta) \]

So what does this say about the optimal path of consumption in the presence of discounting? I says that the marginal utility of consumption must be rising. So therefore consumption must be falling along the optimal path.

Now imagine that we have an interest rate to contend with (and forget about discounting for a second): Whatever we don't spend will grow in value at an interest rate r.

Again consider some allegedly optimal path of consumption. Suppose that I were to move one unit of consumption from period 0 to period 1

would lose: \( U'(C_0) \)

would gain \( U'(C_1) (1+r) \)

Suppose that these two were not equal -- then clearly you were not on the optimal path. So the condition for being on the optimal path is
\[ U'(C_0) = U'(C_1) (1+r) \]

So what has to be happening to consumption in this case? The marginal utility must be falling -- so consumption must be rising.

So now say that I want to characterize the optimal path of consumption in the case where I have both an interest rate r and a rate of time discount \( \Theta \). Clearly the first order conditions relating every two adjacent periods' consumption will be:
\[ \frac{U'(C_t)}{U'(C_{t+1})} = \frac{1+r}{1+\Theta} \]

So what does this tell us? Suppose that \( \Theta \) is greater than r? Then the marginal utility of consumption in period t+1 is higher than the marginal utility of consumption in period t, and so consumption must be falling. What if r>\( \Theta \)? What if they are equal? So interest and discounting work against each other.

If we did know the exact form of the utility function, we could go further. For example, if we know that the utility function is of the CRRA form
then \( U'(C) = C^{1-\sigma} \)

and so the first order condition can be re-written

\[
\frac{C_{t+1}}{C_t} = \left( \frac{1+r}{1+\theta} \right)^{\frac{1}{\sigma}}
\]

Before we discuss the interpretation of this first order condition, we can derive a similar one in continuous time.

To re-write the first order condition with CRRA utility in continuous time:

First note that for small values of \( x \), the approximation \( \ln(1+x) \approx x \) (or alternatively, \( 1+x \approx e^x \)) is fairly accurate.

So for \( 1+r \) we write \( e^r \), and same for \( \Theta \). [being completely accurate, the \( r \) that we use in continuous time, the "instantaneously compounded" interest rate, is not exactly equal to the \( r \) used in discrete time.]

so we can rewrite the first order condition as

\[
\frac{C_{t+1}}{C_t} = \left( \frac{e^r}{e^\theta} \right)^{\frac{1}{\sigma}} = (e^{r-\theta})^{\frac{1}{\sigma}}
\]

re-write this allowing the unit of time used to be a parameter:

\[
\frac{C_{t+\Delta}}{C_t} = (e^{(r-\theta)\Delta})^{\frac{1}{\sigma}}
\]

where if \( \Delta t=1 \) then we have the previous equation.

define as the time derivative of consumption: \( = dc/dt \).
\[
\dot{c} = \lim_{\Delta t \to 0} \frac{c_{t+\Delta t} - c_t}{\Delta t}
\]

Thus the growth rate of consumption is given by

\[
\frac{\dot{c}}{c} = \lim_{\Delta t \to 0} \frac{c_{t+\Delta t} - c_t}{c_t \Delta t} = \lim_{\Delta t \to 0} \frac{c_{t+\Delta t} - 1}{\Delta t} = \lim_{\Delta t \to 0} \frac{\left(e^{(r-\theta)\Delta t}\right)^{1/\sigma} - 1}{\Delta t}
\]

The numerator and denominator of the last expression are both zero when \(\Delta t\) is zero, so we apply L'Hopital's rule, taking derivatives of top and bottom with respect to \(\Delta t\):

\[
\frac{1}{\sigma} \left( (r-\theta)^{-1} \right) \left( e^{(r-\theta)\Delta t} \right) \left( e^{(r-\theta)\Delta t} \right) = \frac{1}{\sigma} (r-\theta)
\]

Evaluating at \(\Delta t=0\), we end up with

\[
\frac{\dot{c}}{c} = \frac{1}{\sigma} (r-\theta)
\]

**Interpretation of the FOC**

In both discrete and continuous time, the FOC says the same thing: the rate at which consumption should fall or grow depends two things: first, the difference between \(r\) and \(\theta\); and second on the curvature of the utility function.

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Completing the solution

Often all we need to look at is the first order condition. But if we want to complete the
solution to the lifetime optimization problem, we can. The FOC tells us how consumption in adjacent periods compares. So given one value of consumption (say, consumption in the first period), we can figure out consumption in all periods -- that is, the entire path of consumption.

[Note, by the way, what will happen to the FOC if r changes over time. This condition would then have to be re-written with r(t) in it, but would be otherwise the same.]

From here, it is simply a matter of finding the value of consumption in the first period that satisfies the budget constraint.

Completing the solution is easiest in the case of continuous time where we let the time horizon (i.e. T) be infinite. Note that there are some technical problems that can crop up in considering infinite time as opposed to just letting T be very large. For example, we can’t impose the “no dying in debt” condition (A(T) = 0), and instead have to impose a different condition (often called the “no Ponzi game condition” that I will not discuss here. For our purposes, it is sufficient to state that the infinite PDV of consumption has to equal the infinite PDV of wages.

Consider a simple case where \( w(t) = 1 \) for all \( t \). Utility is CRRA, \( \theta \) and \( r \) are given.

The first order condition for consumption growth can be integrated to give

\[
\frac{1}{r} = \frac{c(0)}{r + (1/\sigma)(\theta - r)}
\]

Integrating….

\[
c(0) = 1 + \frac{1}{\sigma} \left( \frac{\theta - r}{r} \right)
\]

From this we see

- If \( \theta > r \), then initial consumption is above 1
- The bigger is \( \sigma \), the closer is initial consumption to 1.
**Some Open Economy Applications**

We can use the two period model of consumption to draw a helpful picture. Suppose that we graph the interest rate on the vertical axis, and the level of (first period) saving on the horizontal, with zero somewhere in the middle of the horizontal axis. What is the relation?

Obviously, the position of the curve will depend on the values of $Y_1$ and $Y_2$. (as well as the parameters of the utility function). The bigger is $Y_1$ and the smaller is $Y_2$, the higher will be saving at any given interest rate.

But what about the shape of the curve overall?

We know that if saving is negative, then an increase in the interest rate will raise the amount of saving – we know this because in this case the income and substitution effects are aligned. For zero saving, we also know that the curve is upward sloping. But for positive saving, we don't know – the curve may well bend backward.

Question: what determines the degree to which the curve can bend backward? Answer: the degree of risk aversion!

Why? The degree of risk aversion tells us how the person trades off smoothing of consumption for taking advantage of the interest rate to get more consumption in a later period. If a person is very risk averse, then he wants very smooth consumption. In this case, the curve will end up bending backward.

Now suppose that we have a two-period world, and we are thinking about a country, rather than an individual.

Quick review of open economy national income accounting:

From this we derive the standard national income accounting equation

$$Y = C + I + G + NX$$

one problem: is $Y$ GDP or GNP?

The answer is that we can make it either one; as long as we define imports and exports appropriately.

In fact, for (almost) all of this course, the distinction will not matter. When we think about capital flows, we will be thinking not about portfolio investment or foreign direct investment (FDI) but rather about debt (denoted B). In this case, there will be no foreign ownership of factors of
production, and so GDP and GNP will be the same.

\[ Y = C + I + G + NX \]

\[ Y - C - G = \text{national saving} = I + NX \]

\[ (Y - T - C) + (T-G) = \text{national saving} \]

private saving + gov't saving = national saving = I + NX

Define \( B_t \) as net foreign assets at time \( t \).

The Current Account is the change in net foreign assets. It is equal to \( NX \) plus interest on the assets we hold abroad, minus interest on the debt that we owe foreigners.

In discrete time:  
\[ CA = B_{t+1} - B_t = rB_t + NX_t \]

In continuous time:  
\[ CA = rB + NX \]

So for our thinking about capital flows between countries, there are going to be a variety of assumption that we can make about the different pieces.

**Nature of openness** (for this course, the only type of openness we will think about is capital flows.):

*closed economy*: NX is zero; \( r \) is endogenous.

*small open economy*: \( r \) is exogenous and fixed at \( r^* \), the world level, which is exogenous; NX is endogenous.

*large open economy*: economy is large enough to affect the world interest rate, so \( r=r^* \), but \( r^* \) is endogenous. Also, if this is a two-country world, then NX = -NX*.

Well: a person saving in the first period and consuming more than his income in the second period is exactly equivalent to running a CA surplus in the first period and a CA deficit in the second period. [even though the world only lasts for two periods, we can think of the requirement that people do not die in debt as meaning that \( B_3 = 0 \).]

There is another way that we can think about this same issue, which is more international.

Suppose that there is no trade between countries. Then, since there is no government, \( W=C \) in both periods.
Note that this is not just a case of liquidity constraints in the standard sense. Rather, since everyone is identical, there will be no borrowing or lending. But (key observation): there can still be an interest rate! We think of the interest rate as being the level that clears the market for loans – which will clear at the level where there neither borrowing or lending. This is called the “Autarky interest rate”

To figure out the Autarky interest rate, we can just go back to the first order condition, but now we know that consumption has be equal to $Y$, and so we can just substitute it:

$$U'(Y_1)/U'(Y_2) = (1+r)/(1+\Theta)$$

Now, here is the big result:

$$\Rightarrow$$ If the autarky interest rate is lower than the world interest rate, then the open economy will run a current account surplus in the first period. And if the autarky interest rate is higher than the world interest rate, then the economy will run a current account deficit in the first period.

Intuitively, this is pretty obvious. We can also show it graphically

[The autarky interest rate is what arises in the closed economy version of our model. It is the place where the curve derived above crosses zero. So we can also see here the result about the interest rate!]

Large Open Economy model

Now we can do a large open economy model, making $r (= r^*)$ endogenous. We just draw two versions of the saving vs interest rate diagram that we derived above, and look for the interest rate where saving in one country is the negative of saving in the other. etc.

—

Intuition building problem:

Let's look at the large open economy model with an infinite number of periods, instead of just two.

Let's think about two equally sized open economies. Equally sized in the sense that they have the same endowment income.

$$Y_{1,t} = Y_{2,t} = Y \text{ for all } t$$
we forget about G and I

$\theta_1 < \theta_2$

Two countries start with $B=0$.

What will the equilibrium look like? The key to figuring this out is to realize that the interest rate cannot remain constant! (At least if we assume that consumption can't be negative).

$$\Rightarrow$$ In the long run, we know that the interest rate will be equal to the $\theta_1$. We can trace out the path of interest rates and net assets pretty easily (at least graphically!).

The PIH and the LCH

the model just presented in very standard. The PIH and LCH are two ways of making the same point.

Permanent Income Hypothesis

Developed by Milton Friedman

Rather than focusing on the whole life cycle, the PIH thinks about shorter period changes in income.

The PIH starts by separating income into two parts

$$Y = Y^P + Y^T$$

(note, could have used $Y-T$ here...)

permanent income is the part of income that you expect to persist into the future, sort of like your average future income. Transitory income is the other part of income - the part that is different from the average (note that it can be positive or negative in a given year).

Take a person with a job. Their permanent income is their salary. If in some year they get a bonus, or if in some year they have a smaller salary for some reason, that is positive or negative transitory income.

Think about the following two changes in my income. One month I get a letter saying that I have won $1000 in the lottery. Is that a change in my permanent or transitory income? What about if I get a letter from the dean saying that my salary is higher by $1000 a month?
How will my consumption change in each of these scenarios?

This was Friedman's insight. Your consumption should just depend on your permanent income. To the extent that transitory income is different from permanent income, you will just use your saving to make up the difference.

Let's take an example: suppose you looked at two people, both of whom earned the same amount -- say $100,000 in a given year. One is a businessman, for whom this is the regular salary. The other is a farmer, who has very unstable income, and for whom this was a good year. Which should have higher saving?

So in the PIH, the consumption function is roughly,

\[ C = \alpha Y^p \]

Now we can go back and see how the PIH explains the facts about the consumption function that Keynes failed on.

First think about the long run: over the long run, when income increases, this is clearly a change in permanent income. So consumption and saving will just be constant fractions of income in the long run.

What about in the short run (or looking across households)?

What would you see if all of the variation in income that you looked at was transitory? Then there would be no relation between \( C \) and \( Y \) -- the short run consumption fn would be flat. What if some of the variation were transitory and some permanent? Then you would see what is present in the data. (see homework problem).

We can also give this result an econometric interpretation. Consider a regression of

\[ C = \alpha + \beta Y^p \]

the true value of the parameter \( \beta \) is one. But when we run this regression, we use \( Y \) instead of \( Y^p \) on the right hand side. So the RHS variable is measured with error. There is attenuation bias which biases estimated \( \beta \) toward zero.

One current issue in macro is what Friedman meant -- or what is the truth -- about how far into the future one should look in thinking about "permanent" income. Is it for the rest of your life? For your life and your children's lives? Or is it for some shorter period, like the next 5 years? This will turn out to be important in some of the questions we look at below.
Life Cycle Hypothesis

Due to Franco Modigliani\(^6\)

One direction to go with the analysis of consumption presented above is to look more realistically at what determines saving of people in the economy. Lifetime budget constraint is:

\[
\sum_{t=0}^{T} \frac{C_t}{(1 + r)^t} = \sum_{t=0}^{T} \frac{W_t}{(1 + r)^t}
\]

Now think about your income over the course of life (where we start life at the beginning of adulthood). The biggest thing that you will notice is that there is a big change at retirement -- your income goes to zero.

[picture]

Now think about your preferences. We know that in you are going to want to have smooth consumption -- for example in the case where the interest rate is equal to the discount rate, you will want constant consumption.

[picture]

What is the relation between the income and the consumption lines? Well, if the interest rate is zero, then the areas under them have to be the same. [that is, the sum of lifetime income has to be the same as the sum of lifetime consumption]. If the interest rate is not zero, it is a little more complicated -- what matters is the present discounted value of income is equal to the PDV of consumption.

What does this model say about a person's assets over the course of life?

[picture]

The LCH is also concerned with the total wealth of all of the people in the economy. Why is this so important? Because, for a closed economy, the capital stock of the economy is made up of the wealth of the people in the economy. [and, as you will see when we look at growth, the capital stock is really important].

\(^6\) Once, when asked exactly what the difference was between the LCH and the PIH, Modigliani replied that when the model fit the data well it was the LCH, and when it didn't it was the PIH.
We can see the aggregate amount saved in the economy by just adding up each age group’s saving or dissaving, multiplied by the number of people who are that age. What does this say should be happening to the saving rate of the US as the population ages?

We can also see the effect of social security on saving or total assets in the economy. Social Security lowers income during the working part of life, but raises it during the retirement part of life. So it lowers the saving rate (and level of wealth) at any given age.

[note -- we will talk about the empirical implications of this model and how well they stand up later.]

---

**Income growth and Savings in the Life Cycle model**

How does the growth rate of income affect the saving rate in the life cycle model? Specifically, if we compare two countries that have the same $\theta$ and $r$, and the same age structure, but different growth rates of wage income, which will have higher saving rate.

Answer: it depends on the *form* of income growth. Two cases to look at.

1) Suppose that the shape of the life cycle wage profile is the same in the two countries (it could be flat, or hump shaped, or whatever). Then in the high growth country, the growth rate of wages between successive generations must be larger. This means that if we look at a cross section of the population by age, the growth rate of aggregate wages will be reflected in it, i.e. the youngest people will have relatively higher wages in the high growth country.

2) Suppose that the cross sectional profile of wages in the two countries is the same. Then any individual's lifetime wage profile will reflect this aggregate growth; in this case, people in the high wage growth country will have rapidly growing lifetime wage profiles.

(Of course there could be a mixed case in between 1 and 2 as well)

Cases 1 and 2 yield very different results.

Case 1: here, the lifetime profile of the saving rate is unaffected by growth. The aggregate saving rate is just a weighted average of this, where the weights depend on the number of people and their income. Since young do saving and are richer when growth is higher, higher growth will raise the aggregate saving rate!

Case 2: Now, higher growth affects the saving rate. Specifically, it lowers the saving rate of the young. It also means that working age people (who are saving) earn more than did old people (who are dis-saving) – the effect which we saw in case 1 tends to raise the saving rate. For reasonable parameters, the lowering effect dominates, so higher income growth lowers the saving rate.
Which case is right? Probably 2 is closer to the truth. For example, wage profiles do not depend on aggregate growth rate of income.

Ricardian Equivalence

We will now talk about some of the implications of the optimal consumption/saving models that we have discussed. Later we will look at more direct empirical evidence.

The most controversial implication is the so-called Ricardian Equivalence proposition (which was mentioned, and dismissed, by David Ricardo, and was given its modern rebirth by Robert Barro).

Consider the effect of changes in the timing of taxes. To do so, let's look at the simplest model with taxes, one with just two periods.

Let $T_1$ and $T_2$ be taxes in the first and second periods. Lifetime budget constraint is now:

$$C_1 + \frac{C_2}{1+r} = Y_1 - T_1 + \frac{(Y_2 - T_2)}{1+r}$$

Now consider a change in tax collections that leaves the present value of tax collections unchanged:

$$\Delta T_1 = -Z \quad \Delta T_2 = (1+r)Z$$

For example, if $Z$ is positive (the usual case that we will think about), this would mean that we were cutting taxes today, and raising them in the future. What does this do to the budget constraint?

$$C_1 + \frac{C_2}{1+r} = Y_1 - \{T_1 - Z\} + \frac{(Y_2 - \{T_2 + (1+r)Z\})}{1+r}$$

You can see that the $Z$'s will just cancel out, and the budget constraint is left unaffected. What
about savings, though? Since the budget constraint has not changed, first period consumption will not change. But saving of the people in this economy is equal to

\[ S = Y_1 - T_1 - C_1 \]

So if we reduce taxes by \( Z \), we should raise saving by the same amount. So does the capital stock go up by \( Z \)? No: because the government is going to have to borrow to finance its tax cut. In fact, it is going to have to borrow exactly \( Z \) (or, if it was running a deficit already, it will have to borrow \( Z \) more dollars).

The government will issue bonds, paying interest \( r \), and people will hold them instead of capital -- so the amount of capital will not change. (just like giving people a piece of paper with "bond" written on one side and "future taxes" written on the other.).

Notice that although people who hold the bonds think of them as wealth, as far as the economy is concerned they are not "net wealth," since they represent the governments liabilities, which will in turn be paid by the people.

This is essentially all there is to the Ricardian Equivalence idea.

-- idea has generated a huge amount of discussion among economists.

-- natural application is the explosion of the US government debt in the 1980's and again in the 2000's. One way to look at it is:

\[ Y = C + I + G + NX \]

\[ Y - C - G = \text{national saving} = I + NX \]

\[ (Y - T - C) + (T - G) = \text{national saving} \]

\[ \text{private saving} + \text{gov't saving} = \text{national saving} = I + NX \]

Ricardian equivalence says that if we cut \( T \), it will lower gov't saving, but raise private saving by an equal amount.

-- Can also look at Ricardian equiv in the life cycle model....

-- Similarly, in PIH, tax cuts and increases are just transitory shocks; they do not affect permanent income, and so do not affect consumption.

-- Note that Ricardian equivalence is about the timing of taxes -- it does not say that if the government spending increases this should have no effect on consumption. That is, Ric Equiv says that you care about the present value of the taxes you pay. Government spending, either today or tomorrow, will affect this present value, and so affect consumption. For example, if the govt fights a
war today, your consumption will fall, because you will have to pay for the war. But whether the war is tax financed or bond financed will not matter to your consumption today. [but note that the response of consumption will depend on how long you expect the extra spending to last].

Potential problems with Ricardian Equivalence:

-- different interest rates. If the government can borrow for less than the rate at which people can, then gov't debt may expand budget constraint (at least for borrowers).

-- If people are liquidity constrained in first period consumption, then government borrowing will raise their consumption (show in fisher diagram).

-- If people are myopic whole thing doesn't wash. This is probably true, but hard to model.

--If people are life-cyclers, and will not be alive when the tax increase comes along, then their budget constraints will be expanded and they will consume more. Later generations will get extra taxes and consume less. This objection has generated the most debate, and often discussions of Ricardian Equivalence lapse into discussions about intergenerational relations. Before going along this path, we should note that even if this objection were true, most of the present value of any tax cut today will be paid back by people who are alive today; in which case even if there were no relations between generations, Ricardian Equivalence would be mostly true.

The intergenerational argument in defense of Ricardian Equivalence goes: Since we see people leaving bequests to their children when they die, we know that they must care about their children's utility. Now suppose that we take money away from their children and give it to them. Clearly, if they were at an optimum level of transfer before, they will just go back to it by undoing the tax cut (by raising the bequest that they give).

Much ink has been spilled attacking this proposition. For example:

-- Can specify the motive for bequests in a number of ways: if parents get utility from the giving of the bequest, rather than from their children's consumption (or utility), then a shift out in the parent's budget constraint will lead them to consume more of both bequests and consumption today. Slight variation (Bernheim, Shlieffer, and Summers) is that bequests are payment for services (letters, phone calls) from kids. Same result in response to a tax cut.

-- Alternatively, can argue that bequests are not for the most part intentional, but rather accidental. Consider the life cycle model with uncertain date of death. This model will be covered later. When you see it (with all its discussion of bequests, annuities, etc.), remember why it is relevant to the debate about Ricardian equivalence.

-- Interaction of precautionary savings and Ricardian Equivalence (Barsky, Mankiw, and Zeldes, AER.) -- Don't do in lecture -- just do in HW. (precautionary saving will be discussed below).
One more thing to think about with Ricardian Equivalence: What if people were completely myopic, and never expected to pay back their tax cut. Note that if they were following our usual consumption smoothing models, they would still raise their consumption only very slightly in response to a tax cut (since they would spread their windfall out over the whole of their lives). So Ricardian Equivalence is still almost true in such a case: for example, if people had 20 years left to live, and the real interest rate were 5%, and they kept consumption constant, then a tax cut of $100 that they never expected to pay back would increase consumption by approximately $8. This is pretty close to the zero dollar increase predicted by Ricardian Equivalence. By contrast, if one believed in a Keynesian consumption function (where empirically estimated MPC's are in the rough neighborhood of .75), then there would be a $75 increase in consumption.

[butter, of course, if RE were true and the tax cut were perceived to be permanent (due to a cut in government spending), then C would rise by the full amount of the tax cut].

Deep thought:

Suppose that I look at data on the path of consumption followed by some person (or household). What are the characteristics that I can expect to see in it, assuming that the household is behaving according to lifetime optimization model described above.

I want to argue that one of the most important is that the level of consumption will never "jump," by which I mean that it will never change dramatically from period to period. When consumption does change, it will be because of the difference between theta and r.

So if we do observe consumption jumping up or down, what are we to conclude from it?

I will list some possibilities, but it will take us a while to cover them. But you should see in the list that they are all violations of the simple model presented above.

1)Liquidity constraints – we had been assuming that these didn't exist

1)New information – we had been assuming a world with certainty.

2)"non-convex budget sets," specifically things like means tests – we had been assuming these away since we made income exogenous.

Liquidity constraints under certainty:

Let's return to the issue of liquidity constraints that came up when we looked at the two
period model.

Suppose that you have data on the income and consumption of a large number of individuals, over a long period of time. Each individual is assumed to have known in advance (that is, from the beginning of the sample period), what her income would be for the rest of her life. Individuals in this data set chose their consumption to maximize a usual utility function, with \( u'(c) > 0, \ u''(c) < 0. \) They were able to save money at a real interest rate which was exactly equal to their time discount rate. However, they were not able to borrow money at all.

Individuals did not have smooth income. That is, for each individual there was a good deal of year-to-year variation in income. Furthermore, some individuals had income that rose over the course of the time period examined, while others had income that fell over the time period. Remember, however, that each individual knew in advance what the time path of her income would be.

What would you expect the data on consumption by individuals to look like? In particular, discuss the following two points: First, what general statements can you make about what the time paths of consumption of individuals can look like. What sorts of paths can you rule out? What sorts of paths would you expect to see? Second, what will be the relationship between changes in income and changes in consumption experienced by individuals? Are large decreases in consumption going to occur in the same years as large decreases in income? Will large consumption increases come in the same years as large income increases?

1) consumption can never jump down. When \( \theta = r \), consumption can never fall at all.

2) when consumption jumps up, it must be the case that assets are zero. If \( \theta = r \), then it must be the case that assets are zero when consumption rises at all.

Uncertainty

So far, we have looked at consumption only in a certainty framework. We now look at the effects of uncertainty.

Let's start by reviewing what we mean by expectation. Let \( x \) be a random variable, with probability density function \( f(x) \). Then the expectation of \( x \) is

\[
E(x) = \int_{-\infty}^{\infty} x f(x) \, dx
\]
The relation between the actual realization of a random variable $x$ and its expectation can be written as

$$x = E(x) + \varepsilon$$

where $\varepsilon$ is a random variable with mean zero.

First cut at uncertainty: lifespan uncertainty

In our presentation of the life cycle model, we assumed that the date of death was known. In reality, of course, there is a good deal of uncertainty. To incorporate this into the LC model, we apply the insight that, if you are not alive, you get no utility from consumption.

Let $P(t)$ be the probability of being alive in period $t$. Then an individual maximizes

$$\max \sum_{t=0}^{T} P(U(C_t))$$

$$\sum_{t=0}^{T} (1 + \theta)^t$$

note that we still allow for $\Theta$ to measure pure time discount.

Consider the problem of a person who may die over period $0..T$. Assume that there is no advanced warning.

The formal problem is

$$\max \sum_{t=0}^{T} P(U(C_t))$$

s.t.  

$$A_t = (1+r)(A_{t-1} + W_{t-1} - C_{t-1})$$

$$A_t \geq 0 \quad \text{for all } t$$

$$A_0 \text{ given}$$

Note that here, $W$, $C$, and $A$ are the paths of wages, consumption, and assets that the person will have if they are alive. That is, since the only uncertainty in this model is when you will die, and that uncertainty is not resolved until it happens, you might as well plan out your whole conditional paths of consumption and assets from the beginning (put another way: no new information arrives until it is too late to do anything about it).

Note that the constraint on assets differs from before: as before we say that you cannot die in debt. With certain lifespan, this implies that you have to have zero assets at period $T$. But now, it implies
that you have to have positive assets at all periods!

Maximization problems of this form are generally unpleasant (it will be discussed a little below, when we look at the “Buffer Stock” model of saving). To get around it, assume that we are looking at an elderly person with no labor income, who has only some initial stock of wealth. Such a person would never let wealth become negative, because then she would have zero consumption for the rest of her life.

We set up the lagrangian:

\[
L = \sum_{t=0}^{T} \frac{P_t U(C_t)}{(1+\theta)^t} + \lambda \left( A_0 - \sum_{t=0}^{T} \frac{C_t}{(1+r)^t} \right)
\]

The first order condition relating consumption in adjacent periods is

\[
\frac{U'(C_{t+1})}{U'(C_t)} = \frac{1+\theta}{1+r} \frac{P_t}{P_{t+1}}
\]

we can re-write the second part of the right hand side as

\[
\frac{P_t}{P_{t+1}} = \frac{P_t + (P_t - P_{t+1})}{P_{t+1}} = 1 + \frac{P_t - P_{t+1}}{P_{t+1}} = 1 + \rho_t
\]

where small \(\rho_t\) is the probability of dying in a given period conditional on having lived to that age \([\rho_t = (P_t - P_{t+1})/P_t]\)

The FOC is

\[
\frac{U'(C_{t+1})}{U'(C_t)} = \frac{(1+\theta)(1+\rho_t)}{(1+r)} \approx \frac{(1+\theta+\rho_t)}{(1+r)}
\]

Note that the path of consumption that we are solving for here is the path that the person will follow if she is alive.

So the probability of dying functions just like a discount rate in this case.

------------------------------------

A note of realism: Obviously, the probability of dying rises with age.
Empirically, the probability of dying in old age turns out to conform very closely to a log-linear specification:
\[
\ln(\rho) = \beta_0 + \beta_1 \text{age}
\]

This regularity is known as "Gomperetz's rule"

What will consumption paths of people look like given that this is true?

Suppose that initially, \(\theta + \rho < r\). Then consumption should be rising. But over time, \(\rho\) will rise, and consumption will begin to fall. So there should be these hump-shaped paths of consumption.

Note, by the way, that even though I said that "new information" can be one of the reasons for a jump in consumption, this model with uncertain lifespan does not get you any jumps in consumption – since when the new information arrives there is nothing that you can do about it!

**Annuities**

The person faced with the above problem will almost certainly die holding assets. Only if she lives as long as was remotely possible ex-ante will she die with zero wealth. Assuming that she does not value leaving a bequest, what could make her better off? Answer: an annuity.

Consider a cohort of people with a probability of dying \(p\), and some market interest rate, \(r\). Suppose that a company makes a deal with each person, saying: "Give me your money, and I will pay you some rate of interest \(z\), but if you die before next year I will get to keep your money." What would \(z\) have to be such that the insurance company earned zero profits?

\[
\text{Pays: } (1+z)(1-p) = (1+r) \\

z = (1+r)/(1-p) - 1 = (1+p+r) -1 = r+p
\]

An annuity is an example of such a contract. You give the company money, and they pay you a yearly payment until you die. (Actual annuities are not like the ones described here, in that they pay the same rate of interest each year, rather than paying out at a higher rate as the probability of dying rises).

What is the consumption path of an old person with access to an annuity? The FOC is just

\[
\frac{U'(C_{t+1})}{U'(C_t)} = \frac{(1+\theta+p)}{(1+r+p)}
\]
so in the case where ρ=r, the person would have flat consumption even though her probability of death was rising.

The Value of Being Alive vs. Dead and its implications for Convergence of Full Income

(very simplified discussion of Murphy and Topel, NBER 11405, and Becker et al, “The Quantity and Quality of Life, AER March 2005).

The starting point for estimates of the utility of being alive vs. dead is people’s willingness to trade off risks of death for money. This can be seen in e.g. the wage premium required to get an individual to take a risky job, or the willingness of people to pay for safety features of a product. We generally look at the effects of small changes in the probability of death, but to make such measures useful, we blow them up to the “value of a statistical life.” For example, if a person is indifferent between paying $1000 and taking a 1 in 10,000 risk of death, then the value he is putting on a statistical life is $10,000,000. Estimates of the value of a statistical life in the US are around $6,000,000.

We consider a very simple setup. An individual has constant mortality probability ρ. He never retires, and has constant wage w. The interest rate r and time discount rates θ are equal and greater than zero. Finally, there is an annuity market, so that the interest rate that the individual can earn on his savings is r + ρ. These conditions deliver the result that the individual will want flat consumption, and since he is born with zero assets, he will just consume his wage at every instant.

The individual has CRRA utility with coefficient of relative risk aversion σ. In addition, the individual has utility α just from being alive. His instantaneous utility function is

\[ u = \frac{e^{\frac{1}{1-\sigma}}}{1-\sigma} + \alpha \]

Now, consider a case in which the individual has the opportunity to trade a very small risk to his life for more money. For example, he can spend $500 more to take a safe flight vs. a risky one. We consider a trade between life and risk that leaves the individual indifferent. Let ε be the probability of dying, and let x be the amount of extra consumption that he gets. Since x is small, it does not affect the marginal utility of consumption (it doesn’t matter if we imagine him consuming it all at once or spreading it out over the rest of his life.) The cost in terms of expected life utility lost is

\[ \mathcal{E} \int_{0}^{\infty} e^{-(\rho + \theta) t} \left( \frac{w^{1-\sigma}}{1-\sigma} + \alpha \right) dt = \mathcal{E} \frac{\left( \frac{w^{1-\sigma}}{1-\sigma} + \alpha \right)}{\rho + \theta} \]
The gain is the marginal utility of consumption, which is \( w^{-\sigma} \), multiplied by the extra consumption, \( x \). Putting these together,

\[
\frac{x}{\varepsilon} w^{-\sigma} = \frac{\left( \frac{w^{1-\sigma}}{1-\sigma} + \alpha \right)}{\rho + \theta}
\]

The term \( x/\varepsilon \) is called “the value of a statistical life.”

We can rearrange this to solve for alpha

\[
\alpha = \frac{x}{\varepsilon} (\rho + \theta) - \frac{w^{1-\sigma}}{1-\sigma}
\]

From here, we can just plug in numbers.
I use the following (mostly from Becker et al.)

\[
W = c = $26,000 \quad \text{this is GDP per capita in the US}
\]

\[
x/\varepsilon = $2,000,000 \quad \text{this is on the low end of estimates for the US. (This is roughly what is implied by Becker et al.’s formulation)}.
\]

\[
\sigma = .8 \quad \text{This is their reading of the literature. It seems too low to me, but no one has a good estimate}
\]

\[
r = \theta = .03
\]

\[
\rho = .02 \quad \text{(this gives a 50 year life expectancy)}
\]

Putting these together gives a value of \( \alpha = -8.81 \). Becker et al., using a fancier approach, get a value of -16.2.

In what follows below, I will use their value.

Given a value of alpha, we can ask at what level of consumption an individual is indifferent between being alive or dead. That is, setting utility to zero

\[
0 = \frac{c^{1-\sigma}}{1-\sigma} + \alpha
\]
\[ c = (-\alpha (1-\sigma))^{1/(1-\sigma)} = \$357 \]

[Note: the level of consumption that gives indifference between being alive and dead is incredibly dependent on $\sigma$. To demonstrate this, I did the following. Using the above setup, including $\sigma = 0.8$, I chose the value of life so that I replicated Becker et al.’s value of $\alpha$ (this involved setting the value of life to around $1.5$ million). This then replicated their value of the indifference level of consumption. Holding the other values of the parameters constant, I then changed $\sigma$ to 3. The implied value of the indifference level of consumption is roughly $9,900!$ In fact, if $\sigma$ is 10 (admittedly an unreasonable value), then the break even level is around $18,000$, implying that being a grad student is no better than being dead!]

The intuition for this large effect of $\sigma$ is that when $\sigma$ is large, the marginal utility of consumption falls rapidly with the level of consumption. If sigma is big, then it means that reductions in consumption raise the marginal utility of consumption a lot, so sufficient reductions in consumption very rapidly get you to the point where utility from being alive is zero.

**Implications for Economic Growth**

We usually look at economic growth by looking at growth rates of GDP per capita. But if people get utility from being alive as well as consumption, we should consider their “full income.” (Note: we don’t look at growth by looking at growth of utility. Why not? Because utility is not observable.)

Consider an indirect utility function of an individual with annual income $y(t)$ and survival function (probability of being alive in year $t$) of $S(t)$

\[
V(Y, S) = \max \int_{0}^{\infty} e^{-\rho t} S(t) u(c(t)) dt
\]

s.t.

\[
\int_{0}^{\infty} e^{-\rho t} S(t) y(t) dt = \int_{0}^{\infty} e^{-\rho t} S(t) c(t) dt
\]

Note that we are assuming a perfect annuity market, so that expected lifetime income is equal to expected lifetime consumption. Define $Y$ as the PDV of expected lifetime income. So the indirect utility is a fn of $Y$ and $S$.

Consider a country at two points in time, with lifetime incomes $Y$ and $Y'$ and similar survival
functions $S$ and $S'$. We are interested in the extra income that we would have to give the person so that he would have the same utility he had in the second period, but with the mortality rates observed in the first. Call this extra income $W(S,S')$

$$V(Y' + W(S,S'), S) = V(Y', S')$$

The growth rate of full income is the change in actual income plus this imputed change in income (I think that this is called “equivalent variation”)

$$G = \frac{[Y' + W(S,S')]}{Y} - 1$$

(a few adjustments, not discussed here, have to be made to turn this from PDVs into annual income growth).

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<tbody>
<tr>
<td>Poorest 50% of countries in 1960</td>
<td>41</td>
<td>896</td>
<td>64</td>
<td>3092</td>
<td>1456</td>
<td>3.1%</td>
<td>4.1%</td>
</tr>
<tr>
<td>Richest 50%</td>
<td>65</td>
<td>7195</td>
<td>74</td>
<td>18162</td>
<td>2076</td>
<td>2.3%</td>
<td>2.6%</td>
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So poorest countries get big income growth boost

Other things to do with this:

What is implied risk behavior as income gets really low?

Relate this to Stone Geary – very different implications.

**Precautionary saving**

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We now ask a different question about uncertainty: how does uncertainty affect behavior?

It is easy to show that uncertainty makes you worse off. In the single period case, we know that, if $c$ is some random variable:

$$E(u(c)) < u(E(c))$$

So, if you could pay to reduce uncertainty, you would do so (for example, buying insurance). But now we want to ask a different question: does uncertainty raise saving?

Let's examine the question in a simple two period model. For simplicity, we will assume that there is no discounting and no interest rate. In period 1 you get income $Y$ and consume $C_1$. In period 2, you participate in some lottery: with probability $.5$ you get some amount $L$ given to you. With probability $.5$ you have the same amount taken away from you. So your consumption in period 2 is

$$C_2 = Y - C_1 + L \text{ with probability } .5 \text{ and}$$

$$Y - C_1 - L \text{ with probability } .5$$

Your only choice variable is $C_1$. We will want to answer the question: how does saving (or first period consumption) change when the size of the lottery changes (that is, as uncertainty is increased).

Expected utility is just:

$$E(U) = U(C_1) + E(U(C_2)) \text{ (Notice that we don't need to put an expected value symbol in front of } C_1,$$

$$\text{since it is known).}$$

$$= U(C_1) + .5*U(Y - C_1 + L) + .5*U(Y - C_1 - L)$$

We maximize this by taking the derivative with respect to $C_1$ and setting it equal to zero:

$$0 = U'(C_1) - .5*U'(Y - C_1 + L) - .5*U'(Y - C_1 - L)$$

The optimal value of $C_1$ is the value that solves this equation. That is, the equation implicitly tells us $C_1$ as a function of $L$. What we care about is how $C_1$ changes as $L$ changes. To find this derivative, we use the implicit function theorem (See Chaining). If we write this as $0 = F(C_1,L)$, the implicit function theorem tells us that:

$$\frac{dC_1}{dL} = - \frac{F_L}{F_{C_1}} = - \frac{-.5[U'(Y - C_1 + L) - U'(Y - C_1 - L)]}{U'(C_1) - .5[-U'(Y - C_1 + L) - U'(Y - C_1 - L)]}$$
Notice that since $U''(C)<0$, the term in the denominator that is in square brackets is positive, and so the whole denominator is negative. So the key term is:

$$U''(Y - C_1 + L) - U''(Y - C_1 - L)$$

When will this term be positive? When $U'''(C) > 0$. In this case, the whole expression is negative, and so

$$\frac{d C_1}{d L} < 0$$

in which case there is precautionary saving.

Assuming $U''' > 0$ is what we have been doing in drawing the marginal utility curve as [picture]

The intuition is that spreading out consumption in the second period raises the expected value of the marginal utility of consumption in the second period. This means that in taking part of consumption away from the first period and moving it to the second period (ie by saving more), you can, in expectation, get higher marginal utility.

This result about $U''' > 0$ being necessary for precautionary saving is not unique to the two period model -- it holds generally.

Many of our favorite utility functions, such as log and CRRA, have positive third derivatives, and thus imply precautionary savings.

A utility function that doesn't imply precautionary savings is quadratic utility:

$$U(C) = \beta_0 + \beta_1 C - \beta_2 C^2$$

Here the first derivative is positive (for low enough $C$), the second negative, and the third is zero.

(We know that quadratic utility cannot be globally correct, since it implies that marginal utility becomes negative at some point. But it can still serve as a useful approximation as long as we restrict our attention to a limited range of values of $C$)

We often use quadratic utility for mathematical simplicity, and also sometimes precisely because it gets rid of precautionary savings.
If there is no precautionary savings (and also no precautionary dis-savings), then the economy or consumer is said to display “certainty equivalence.” That is, they act as if future income at every period \( t \) were certain to be equal to \( E(Y_t) \).

\[ \text{Another application of this idea is precautionary childbearing [expand this; point out that positive third derivative is a natural assumption.]} \]

\[ \text{The general way to deal with these sort of uncertainty problems is via dynamic programming, which you will see later in your math course.} \]

3\textsuperscript{rd} approach to uncertainty: Stochastic Income (leading up to Hall’s Euler Equation approach)

Example of a stochastic process and permanent income.

Suppose that we are the consumer, and we have some form of expectations about future income: what is the optimal level of consumption that we should choose. [This example is taken from Abel's chapter in Handbook of Monetary Economics.]

We consider optimal consumption for a person with income that follows a stochastic process:

\[ w_{t+1} = \alpha w_t - + \varepsilon_{t+1} \]

where \( \varepsilon \) is iid mean zero.

Describe what is meant by this stochastic process. If \( \alpha \) is zero, then income is iid with mean \( \bar{w} \). If \( \alpha \) is one, then income is a random walk. More generally, the size of \( \alpha \) tells us how persistent the income process is -- that is, how long a shock lasts.

Consider a person with

1) quadratic utility
2) \( r = \Theta \)
3) infinite horizon

1) will give us "certainty equivalence." That is, consumption will be the same as if there was no uncertainty (even though utility will be lower).

2) will give us flat desired consumption

3) This is for convenience -- things would look almost the same with a "long" horizon.
So for certainty equivalence, we simply have to set consumption today at the level that would be sustainable in expectation. That is, if all future values of $\varepsilon$ were zero.

so the budget constraint is

$$\sum_{s=t}^{\infty} \frac{c_t}{(1+r)^{t-s}} = A_t + E_t \left[ \sum_{s=t}^{\infty} \frac{w_s}{(1+r)^{s-t}} \right]$$

notice that the subscript on consumption is $t$, while on wages it was $s$.

Of the two terms on the right hand side, the first is just assets (ie wealth), and the second is the pdv of future wages -- which we could call "human wealth."

We use the rule that

$$\sum_{x=0}^{\infty} x^r = \frac{1}{1-x}$$

so the left hand side can be replaced with

$$c_t((1+r)/r)$$

so $c_t = (r/(1+r))^t(A + HW)$

(or, in continuous time, $c = r^t(A + HW)$ -- this should be fairly intuitive looking. It just says that if you want to have constant consumption, you should just consume the interest on your wealth, where wealth is actual plus human).

Notice that if $w$ were constant, then the pdv of future wages would be $\bar{w}0/r$ -- but because of our timing assumption, HW would include today's wages, so HW would be $\bar{w}0((1+r)/r)$, and so consumption would be

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\[ c = \frac{r}{(1+r)}A_t + \bar{w}0 \]

Now we can apply our same rule to the right had side of the equation. First re-write \( w_t \) as \( \bar{w}0 + (w_t - \bar{w}0) \)

The sum of expected is just \(((1+r)/r) \* \bar{w}0.\)

As to the other term, from the stochastic process, we know that

\[
E_t (w_{t+1} - \bar{w}0) = \alpha(w_t - \bar{w}0) \\
E_t (w_{t+2} - \bar{w}0) = \alpha^2(w_t - \bar{w}0) \quad \text{etc.}
\]

So we get

\[
\frac{1+r}{r} c_t = A_t + \frac{1+r}{r} \bar{w} + \sum_{s=t}^{\infty} \frac{\alpha^{s-t}(w_t - \bar{w})}{(1+r)^{s-t}}
\]

re-arrange and get:

\[
c_t = \frac{r}{1+r} A_t + \bar{w} + \frac{r}{1+r - \alpha}(w_t - \bar{w})
\]

This says that the amount by which current income affects consumption depends on the persistence of income.
The Euler Equation Approach to Consumption

In the above example, consumption ends up being a function of only current income and current assets, with the MPC out of current income depending on the degree of persistence in the time series (ie on $\alpha$).

The above approach used a particularly simple stochastic process for income (ie an AR(1)). However, income could follow a much more complex stochastic process (for example Autoregressive Moving Average process, called ARMA – you can learn about these in a time series course.) In such a case, to predict future income we need not only know today's income, but also income from several past periods. Since consumption depends on the PDV of income, this would imply that consumption was also a function of not only today's income, but also income from several past periods.

Going even further, there might be things other than past income that were observable today and which also helped predict future income. For example,

- suppose that we are looking at data on an individual. Say that she is a professor. Then her future income growth depends on the number of publications. So her optimal consumption depends on her number of publications.

- Suppose that we have data on people's professions (plumber vs surgeon). Different professions have different wage profiles (flat for plumbers, rising for surgeons). Since the individual knows this, he should take it into account in figuring out the PDV of future wages, and thus this should affect consumption today as well.

- if we are looking at the aggregate economy, there may be pieces of data available at time t that tell us about expectations of future income, for example the value of the stock market or the consumer confidence index. These should affect consumption today as well.

The upshot is that consumption today should be a function of wages today and of lots of things that predict how wages will change in the future. So one could imagine estimating a very complicated "consumption function" that incorporated all of this kind of stuff on the right hand side.

And indeed, people do this. The problem is that it is very hard to interpret what you get in terms of any theory.

Now, suppose that instead of looking at the level of consumption, you look at the change in consumption, that is, $c(t+1) - c(t)$. What do we expect to be the predictive power of all of this stuff
(current income, past income, other stuff) that went into the consumption function? After all, this stuff went into the consumption function in the first place because it helped to predict how income would change over time.

The answer is that under the PIH, this stuff should have no predictive power at all!

So now suppose that you tried to predict consumption in period $t+1$ using consumption in period $t$ as well as other information available at time $t$. The test that Hall proposed is that nothing else available at time $t$ should matter. This turns the usual sort of econometrics that we do on its head: usually, we do empirical work hoping that what we look at will matter (as judged by a $t$ statistic, for example). Here, the theory succeeds if other things do not come in significantly.

More formal description of the Euler Equation approach:

Consider the consumption plan formulated at some date $t$. That is, consider a person at time period $t$ who is deciding on consumption today. We know that, along this consumption path, the relation between $C_t$ and the $C_{t+1}$ is

$$E_t[U'(C_{t+1})] = \frac{1+\theta}{I+r} U'(C_t)$$

where now consumption in period $t+1$ has an expectation sign in front of it because it is not actual consumption in that period, but just what is expected to be consumed.\(^7\)

Note that all of the things on the right hand side are known already at time $t$.

Recall that the relation between the actual realization of a random variable $x$ and its expectation can be written as

$$x = E(x) + \varepsilon$$

where $\varepsilon$ is a random variable with mean zero.

Or in this case,

$$U'(C_{t+1}) = E_t[U'(C_{t+1})] + \varepsilon_{t+1}$$

\(^7\) This first-order condition relating consumption in adjacent periods is also known as the Euler equation, and so this approach to studying consumption is often called the “Euler Equation Approach.”
That is, the actual marginal utility of consumption at time $t+1$ will be equal to the expected marginal utility plus a mean-zero error.

If we assume that $r=\theta$, then we can combine these last two equations to get

$$U'(C_{t+1}) = U'(C_t) + \epsilon_{t+1}$$

This says that the marginal utility of consumption follows a random walk. Of course we can't observe the marginal utility of consumption – we can only observe consumption itself.

Now, assume quadratic utility:

$$U(C) = \beta_0 + \beta_1 C - \beta_2 C^2$$

then

$$U'(c) = \beta_1 - 2\beta_2 C$$

So the above equation becomes

$$\beta_1 - 2\beta_2 c_{t+1} = \beta_1 - 2\beta_2 c_t + \epsilon_{t+1}$$

or

$$c_{t+1} = c_t + \epsilon_{t+1}$$

where the epsilon is slightly differently scaled, but is the same mean zero error term.

So if utility is quadratic, consumption should follow a random walk.

Now, consider what should happen when we regress the change in consumption on all of the “stuff” observed in period $t$ that went into the consumption function for the level of consumption (ie current and past income, consumer confidence, the stock market, papers published, etc.). All these things predicted changes in income, but they should not predict changes in consumption. The reason is that changes in income that they predicted were already taken into account when optimal consumption in period $t$ was calculated.

This is the test of the PIH proposed by Robert Hall. Specifically, suppose that you found some variable that plausibly predicted changes in income, and which also came in significantly when you regressed the change in consumption on it (and was in the information set of people at time $t$ when they made their consumption decisions). This would violate the PIH.
Hall's test of the permanent income hypothesis is so clever because it can be performed by an observer (i.e. the econometrician looking at data) who knows very little about how the consumption decision is being made. For example, we don't have to know anything about what the person's expectations of future income are.

(One of the clever aspects of the Hall approach is that it does not rely on being able to actually estimate the consumption function. The essence of the “Lucas Critique” is that the consumption function should not be stable when the economic environment changes, and so estimating consumption functions went out of style.)

More generally, if r is not equal to theta, or if utility is not quadratic, the implication of the PIH is that consumption should just be predictable by last period's consumption and some constant growth term. If there are any other ways in which consumption is predictable, then the LCH/PIH does not hold.

Empirical evidence on Euler equation results: it turns out that the question gets complicated. Can see Hall for a summary.

An Aside: Tax Smoothing

The discussion earlier of Ricardian Equivalence took the movements of taxes from one period to another as being exogenous. But a related literature has asked what is the optimal pattern of taxes. That is, given that the government has a certain amount of spending that it wants to do, how should it arrange taxes to finance it?

To answer this question we have to think a little harder about the question of how the taxes are collected.

We often assume that taxes were lump sum -- for example, $1 per person. Why is this the easiest assumption to make? In this case, taxes do not have any effect on behavior other than through the budget constraint. (In fact, even if taxes fall on labor income, if the supply of labor is

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8 The beauty of this approach is that, unlike old fashioned estimation of consumption functions, it does not run afoul of the Lucas Critique. Since the Lucas Critique has not yet been presented in this course, this footnote will not be very meaningful the first time you read it. But later in the course it should make sense.
inelastic, then there is still no distortion, so taxes might was well be lump sum.)

In the case of lump sum taxes, we have seen that the timing of taxes may not matter at all. But in the real world, very few taxes are lump sum. Instead, taxes are levied on income or consumption -- that is, you have to pay the tax when you engage in economic activity. We have learned in Micro that such taxes are distortionary.

As an example, consider a tax on wages. We have some supply for labor and some demand for labor, an equilibrium price and quantity, and an associated consumer and producer surplus (note that in this case the "consumer" is the firm, while the "producer" is the worker.). Now the government imposes a tax on wages -- it doesn't matter if this tax is imposed on the worker or the firm, the result is the same. The tax imposes a wedge between what the producer gets and what the consumer pays. In equilibrium, quantity goes down. We can show the areas of consumer surplus, producer surplus, and government revenue.

Note that a little triangle got lost -- this is the loss of efficiency due to the tax. This is the "dead weight loss."

Now suppose we wanted to graph a function that related the loss of efficiency per unit of revenue collected. When the tax rate is zero, revenue is zero. But the marginal revenue from imposition of a tax is high, while the marginal efficiency loss is zero. As the tax rate rises, each rise in the tax rate costs more in efficiency, and earns less revenue. So the cost in efficiency per tax dollar collected is an increasing and convex function (that is, its second derivative is positive).

Now you should see the relation between the government's problem in choosing tax collections over time and the individual's problem of choosing consumption over time. Individual has a concave utility function, and wants to maximize utility. Government has a convex "distortion" function, and it wants to minimize total distortions. In both cases, the optimal thing is to smooth. So in the presence of distortionary taxes, if Ricardian Equivalence holds, gov't wants constant taxes. And all of the Permanent Income type thinking that we are used to doing for consumption should also hold for taxes: for example, suppose that the government has to fight a war. Should it pay for it by taxing, or by borrowing? Clearly, this is a temporary change in it's spending. We know that an individual's consumption should not react much to a temporary change in income; similarly, taxes should not react much to a temporary change in spending.

(Note: We have been assuming that the interest rate is invariant to all of this moving around of taxes. Maybe that is right for a small open economy fighting a war. But for a closed economy, there will be a reaction of the interest rate to the fact that the government wants to borrow a huge amount during a war. So all of the conclusions of this section have to be revised).

Notice that given what we said about tax smoothing by an optimizing government, the random walk property should also apply to taxes: nothing other than today's taxes should predict tommorow's taxes. So if \( r = \Theta \), there should never be any predictable changes in taxes. (See Barro article on this topic).
Precautionary Savings and Stochastic Income

Now we look at a similar problem of solving for optimal consumption when income is stochastic. But this time we are interested in precautionary savings, so we no longer set up the problem with certainty equivalence.

Recall the difference between risk aversion and precautionary savings. Risk aversion says that you are made worse off by uncertainty. This is just due to $u''<0$. Precautionary saving says that you will change your behavior - save more - in response to uncertainty about future income. This requires that $u'''>0$.

For this problem we will use a utility function where $u'''>0$. However, we are not going to be able to use our favorite CRRA utility function, because in this case it is not analytically tractable. Instead we use the Constant Absolute Risk Aversion utility function:

$$U(c) = \frac{-1}{\alpha} e^{-\alpha c}$$

$$u' = e^{-\alpha c} \quad u'' = -\alpha e^{-\alpha c} \quad u''' = \alpha^2 e^{-\alpha c}$$

[What is the difference between CRRA and CARA? Consider the question of how much someone is willing to pay to get rid of a certain risk. For example, how much would you pay to get rid of a risk that consumption will go up or down by $100, each with probability 50%. Say that the base level of consumption is $500 (so that consumption might be either $400 or $600), and you find that a person is willing to pay $30 to eliminate the risk (that is, they are indifferent between $470 with certainty and the lottery of $400 or $600). Now suppose that we raise their base consumption to $1,000 but keep the size of uncertainty constant: their consumption will be either $900 or $1,100. How much will they pay to insure this risk. Under CARA, the answer would still be $30. Under CRRA, it would be less than $30, since the risk relative to consumption has declined. Under CRRA, it would be the case that they would pay $60 to insure a risk of $200 on a base of $1,000..... extend this example?

Notice that a property of CARA is that when consumption is zero, the marginal utility of consumption is finite. By contrast, for CRRA, when consumption is zero the marginal utility of consumption is infinite. Indeed, the CARA utility function is defined for negative consumption, while CRRA is not. This makes CARA suitable (mathematically) for problems like the one we will

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\(^{9}\)Blanchard and Fischer, p. 288-291 and notes from Cecchetti.
do now, where consumption may sometime end up being negative in order to satisfy a lifetime budget constraint. Whether this makes sense economically is a different question. Consumption can't really be negative, but on the other hand, the property of the CRRA utility function that marginal utility is infinite at zero consumption may overstate the importance of that state of the world... even a low probability of infinite marginal utility will dominate optimization. In fact, if consumption gets near zero in advanced countries, other mechanisms kick in... we will address these below.

We will assume that theta=r=0 for convenience. We also assume a finite horizon.

We assume that income follows a random walk:

\[ Y_t = Y_{t-1} + \varepsilon_t \]

Where \( \varepsilon \) is distributed \( \text{N}(0, \sigma^2) \)

Notice that income can become negative.

So the optimization problem at time \( t \) is:

\[
\text{Max } E_t \left[ \sum_{s=t}^{T-1} \frac{1}{\alpha} e^{-\alpha c_s} \right]
\]

where assets and income at time \( t \) are known.

I will state the solution, and then prove that it is correct.

The solution is:

\[ c_t = \frac{1}{T-t} A_t + Y_t - \frac{\alpha (T-t-1)}{4} \sigma^2 \]

We will first check that it satisfies the first order condition. Given zero interest and discount rates, the FOC is just:

\[ U'(c_t) = E_t[U'(c_{t+1})] \]
First we will solve for \( c_{t+1} \) in terms of things that we know at time \( t \).

The evolution of assets if given by

\[
A_{t+1} = A_t + Y_t - c_t
\]

\[
= A_t + Y_t - \left( \frac{1}{T-t}A_t + Y_t - \frac{\alpha(T-t-1)}{4}\sigma^2 \right)
\]

\[
= A_t \left( \frac{T-t-1}{T-t} \right) + \frac{\alpha(T-t-1)}{4}\sigma^2
\]

Now, we write \( c_{t+1} \) in terms of quantities at time \( t+1 \), being very careful to write in the proper time index!

\[
c_{t+1} = \frac{1}{T-t-1}A_{t+1} + Y_{t+1} - \frac{\alpha(T-t-2)}{4}\sigma^2
\]

We can substitute in for assets and income at time \( t+1 \):

\[
c_{t+1} = \left( \frac{1}{T-t} \right)A_t + \frac{\alpha}{4}\sigma^2 + Y_t + \varepsilon_{t+1} - \frac{\alpha(T-t-2)}{4}\sigma^2
\]

\[
c_{t+1} = \left( \frac{1}{T-t} \right)A_t + Y_t + \varepsilon_{t+1} - \frac{\alpha(T-t-3)}{4}\sigma^2
\]

So
\[ c_{t+1} = c_t + \varepsilon_{t+1} + \frac{\alpha}{2} \sigma^2 \]

and

\[ E_t( c_{t+1} ) = c_t + \frac{\alpha}{2} \sigma^2 \]

From the utility function:

\[ U'(c_{t+1}) = e^{-\alpha c_{t+1}} \]

But we care about the expectation of this. Luckily, there is a rule that we can use in this case: Let \( x \) be some random variable distributed normally with mean \( \mu \) and variance \( \sigma^2_x \). Then

\[ E( e^{x} ) = e^{E(x) + \sigma^2_x / 2} \]

Notice that this rule corresponds to Jensen’s inequality: the exponential function is convex, so the expectation of exponential of a random variable is greater than the exponential of the expectation of the random variable (say that five time fast).

For the current problem, \( c_{t+1} \) is a random variable when viewed from the perspective of time \( t \). We already solved for its expectation, and its variance is just \( \sigma^2 \).

\[ E_t(U'(c_{t+1})) = E_t(e^{-\alpha c_{t+1}}) = E_t \left( e^{E_t(-\alpha c_{t+1}) + \frac{\text{Var}(\alpha c_{t+1})}{2}} \right) \]

we know that
So the first order condition checks!

Checking the budget constraint is much easier. We only have to check that if we follow the suggested policy, end-of-life assets will be zero. As usual, T-1 is the last period of life, and so we just check that assets at the beginning of period T would be zero: we substitute into the formula for assets in period t+1 above, letting period t be period T-1:

\[ E_t(-\alpha c_{t+1}) = -\alpha E_t(c_{t+1}) = -\alpha c_t - \frac{\alpha^2}{2} \sigma^2 \]

\[ \text{Var}(-\alpha c_{t+1}) = \alpha^2 \text{Var}(c_{t+1}) = \alpha^2 \sigma^2 \]

So

\[ E_t(U'(c_{t+1})) = e^{-\alpha c_t} = U'(c_t) \]

So the first order condition checks!

Most important result is that expected consumption rises over time. Further, the speed with which it rises depends on the degree of uncertainty: \( \sigma^2 \). This confirms the result of our two-period model of precautionary saving.

We can derive a testable prediction from the model of precautionary saving: Think if you followed a cohort of people over their lives, where each person had uncertain income, but on average there was no uncertainty. What would you expect to see for average consumption of the people in the cohort? -- it would be rising over time.

Now suppose that we grouped people into occupations. Within each occupation, there is an expected wage profile (say: flat for mechanics, rising for brain surgeons, etc.) Should the slope of
the wage profile affect the slope of people's consumption profiles? No, of course. But suppose that occupations differed in their degrees of uncertainty? We could look at people in risky vs non-risky occupations. The model predicts that people in less risky occupations should have lower average rates of consumption growth.

This is a generally applicable point: there are many situations where on average there is not much uncertainty, but where at the individual level there is a good deal of uncertainty. Some examples: health, date of death, success in career, etc. Macro-economists used to ignore this sort of uncertainty, precisely because at the societal level the law of large numbers them go away. But if they affect behavior, then the uncertainty is relevant.

Liquidity constraints under uncertainty:

if $\theta > r$, this leads to buffer stock saving (do more formally?).

Here is the simplest buffer stock model:

$E(w)$ is constant, but there is variance (iid)

- time horizon is infinite (or long)
- $\theta > r$
- liquidity constraint

== if there was no uncertainty and no liquidity constraint, would get a declining path of consumption
== if there was uncertainty but no liquidity constraint, would also get a declining path, adjusted for shocks
== if liquidity constraint but no uncertainty, would get consumption = wages as constrained optimum.

== if both, it is complicated. We have to solve by dynamic programming.

Define cash on hand as $w + rA$ – that is, all that you have available to spend this period. Since income is iid, this is all you can base your decision on. So we are looking for $c$ as a function of cash on hand.

===> handout picture from Deaton book.

To be added (maybe): non-linear budget sets.
In real life, the assumption that income is completely exogenous to consumption choices does not hold. One of the most important feedbacks from consumption to income is in the form of "means tested" government programs, which provide income to you only if you are poor enough to qualify. Examples are Medicaid, Food Stamps, Welfare, financial aid for college, etc.\textsuperscript{10}

33. Consider a world in which people live for two periods. In the first period of life, person $i$ has income of $x_i$. In the second period, all people have income of zero.

Individuals can save money at a real interest rate of zero. They cannot borrow. They are born and die with zero assets. Individuals have log utility, and their time discount rate is zero.

There is a government program which has the following setup: if the sum of your income and your savings (from the previous period, if any) in a given period is less than a cutoff level, $c$, then the government will give you enough money so that you can consume $c$. If the sum of your income and savings in a period is higher than $c$, then the government will not give you anything.

Describe the relationship between the first-period saving rate (that is, saving divided by first period income) and first period income. If you can't get exact solutions, try drawing a picture and discussing how you think it might be done.

\textbf{Alternative forms of the utility function}

We have been assuming time-separable preferences -- so utility at a given time depends on just consumption at that time.

Is this reasonable? Quite possibly not: two people with the same consumption might have different happiness depending on what they were used to.

There is nothing radical about non-time-separability (unlike some other alternatives to the usual utility function) -- when we use time separable preferences it is really only for convenience. One form of time non-separability is durability [think about food or vacations]. If utility from consumption is durable, then the more I consume in period $t$, the lower my marginal utility of consumption is in period $t+1$.

We will look at another form of non-separability called habit formation, in which consuming

\textsuperscript{10} There is also an important feedback at the aggregate level: the more that people in a country save, the higher will be the capital stock, and thus the higher will be wages. We will deal with this feedback extensively when we study growth.
more in a period raises the marginal utility of consumption in the future.\(^\text{11}\)

Consider a utility function like

\[
U(c, z) = \left( \frac{c}{z^\gamma} \right)^{1-\sigma} \frac{1-\sigma}{1-\sigma}
\]

where \(z\) is the quantity of habit.

Notice that habit makes you worse off. If \(\gamma\) is zero, then habit is irrelevant. If \(\gamma=1\), then only the ratio matters. So \(\gamma\) indexes the importance of habits. The closer it is to 1, the more people care about consumption relative to habit. The closer it is to zero, the more people care about the absolute level of consumption. For example if \(\gamma=.5\), then a person with consumption of 4 and habit stock of 4 will have the same utility as a person with consumption of 2 and a habit stock of 1.

Habit in turn evolves according to the level of consumption:

\[
z_{t+1} = \rho c_t + (1-\rho) z\]

The bigger is \(\rho\), the faster habits adjust.

(As an aside: we have been taking “habit” or “what you are used to” to depend on your own past consumption. Another possibility is that it depends on the consumption that you observe around you. This is the idea of “keeping up with the Jonses.”)

This can also be incorporated into a utility function. The classic case was Dusenberry, who formulated the “relative income hypothesis” in an attempt to explain the same facts that Friedman successfully explained by the PIH.)

Anyway, these things are a mess to solve, but they may provide better fit to reality that the function that we use.

Interesting thoughts: do people understand their own future habits? See Keynes on Economic Prospects! Discuss COW. Example of tenure and similar.

\(^{11}\) It is perfectly possible for durability and habit-formation to coexist at different time horizons. For example, if I consume a fine French meal at noon, it lowers the marginal utility of French food at dinner time, because I am still full. But it also raises the marginal utility of French food in future days because I will not longer be full and now have acquired a taste for good food.
New Material to be added:

More formal discussion of optimization in habit formation model. Can do something based on Deaton formulation (see deaton book or Paxson eer): quadratic sub utility (muellbaur form) gives simple first order condition even in stochastic model.

----------------------------------------
Inter-temporal consistency.

Go back to time-separable preferences.

You will see when we you study monetary policy later on, that one of the key problems that faces a policy maker is time inconsistency (example: negotiating with terrorists; nuclear war; monetary policy).

Question: In a world of certainty (which we have been doing so far), would you ever want to re-optimize. That is, if you can make all of your decisions at time one, are you then happy to stick with them?

The classic paper that deals with this problem is Stroz (1956), which I confess I haven't read.

Stroz shows that the only form of discount factors which does not lead to inconsistency of the lifetime utility function is our usual exponential discounting (i.e. dividing utility in period t by \((1 + \theta)^t\) in discrete time, or multiplying by \(e^{-\theta t}\) in continuous.

Any other form of discounting leads to the paradox that the relative importance of two periods' consumption depends on what point in time the problem is being viewed from

[picture]

Some people (such as Angus Deaton in his book) think that Stroz's result shows that any form of discounting other than exponential is irrational, and thus should not be considered.

Recent work by David Laibson takes on this challenge: argues that preferences are not, in fact, exponential, and further that there is loads of evidence of inconsistency.

examples of inconsistency: revisions of plans, diets, Christmas clubs and other forms of hand-tying, etc.
Laibson claims that the natural discount function is hyperbolic: so discount the immediate future a lot, but not much between immediate and far future. [note: “hyperbolic” is not the mathematically correct term for what Laibson posits, but the term has stuck.]

The actual utility function that he uses in his paper is

\[ U_t = E_t \left( u(c_t) + \beta \sum_{\tau=1}^{T-t} \frac{1}{1 + \theta} u(c_{t+\tau}) \right) \]

where \( \beta < 1 \)

hyperbolic discounting means that my (today) self will want to precommit: forced saving, purchase of durables, etc.

Laibson then models a game between different “selves” at different periods.

Notice that the selves will want to pre-commit later selves. For example, I might want to save my money in a way that it can only be taken out with one period’s advance request...

Laibson also argues that the recent decline in the saving rate comes from the ability to get around these sort of precommitment devices.

[add stuff from Thaler?]

[for somewhere: talk about Phillipe Weil & non-expected utility].

**Empirical evidence on LCH/PIH**

In this section I will present a grab-bag of pieces of evidence relating the LCH/PIH. Since the theory is a little amorphous and the data is imperfect, there is never going to be any one test that is completely convincing. Three points to looking at all of these tests: First, get an overall impression of how good the whole approach is. A second goal is to give some examples of how one applies data to the testing of a theory. Third, to see how theory can be twisted around to make it fit the data.

1.1 Tests of the LCH looking at the wealth of old people. If one takes the life cycle model seriously, and doesn’t think that inter-generational relations are important, then one would expect to see people’s assets declining in old age. Since there is uncertainty, we do not expect to see assets hit zero on the day before death, but we still expect to see them going down.
Many people have tried to test the LCH by seeing if old people have negative saving rates. The answer tends to be that they either keep constant assets or they dissave slightly. To justify this with just uncertainty, one would have to posit extreme risk aversion.

1.2 Variation on this test is to ask whether a bequest motive can explain this failing of the pure LCH. Test is to look separately at the behavior of people with and without children, under the assumption that people with children have stronger bequest motives. Result is that there is no difference between the behavior of the two: this is bad news for the intergenerational version of the LCH.

2. Tests of the life cycle model as an explanation for the aggregate capital stock. One point of the LCH is to explain the size of the aggregate K stock. Can test this two ways:

2.1 -- Simulation. Start with realistic parameters for discount rate, interest rate, wage path over course of life cycle, population growth, etc. Then can figure out what each person's wealth should be. Sum this up across people, and compare it to aggregate income [also derived by summing up individual income across age groups]. This gives you a wealth/income or K/Y ratio. Now compare this to what we observe in the economy [roughly 3]. Result: ratio delivered by model is too small. [cite: White?] In other words, the LCH does not explain most of the capital stock.

2.2 (similar in philosophy). Kotlikoff and Summers divide the current stock of wealth into two parts: life cycle wealth and transfer wealth. LC wealth is the difference between the past income and past consumption of people currently alive (inflated at the interest rate, of course). These profile are derived by looking at the current cross-sectional profiles of Y and C, and then having them grow over time at the rates of growth of aggregate Y and C.

Result: LC wealth is only equal to about 20% of total wealth. The rest of wealth must be the result of money passed down by people no longer alive: that is, transfer wealth.

They argue: since LC wealth is not that big, we should not focus on the LC mode.

This brings back to the fore the question of what motivates bequests.

3. Relation between lifetime income and consumption profiles.

Note that we don't expect consumption profile to be flat, since tastes may change over lifetime. For example, when you have kids, you may want to spend more. Similarly, when you are old, you may have less utility from consumption (i.e. can't go on vacations).

The key prediction of the LCH/PIH is that your consumption profile should be invariant to the shape of your income profile. That is, two people with the same discounted lifetime value of income but different patterns in which it is received should have the same consumption profile. For example, an athlete who makes all his money early in life, vs a brain surgeon who doesn't start earning until late in life, but then makes a lot.

This is one of the things that Carroll and Summers look at in their paper. They find that, breaking people down by occupation, the averages income and consumption profiles are, in fact, very similar.
They interpret this as meaning that the LCH/PIH is just wrong.

One counter argument to their results: Suppose that utility is a function of some combination of consumption and leisure. What is the price of leisure? It is just the wage. So when wages are high, then leisure is expensive, and so people consume less of it. This may raise the marginal utility of consumption, and so people will consume more (since, ignoring interest and discounting, they are setting the marginal utility of consumption equal in every period.). Key the sign of $d^2u / dc dL$ (homework problem).

4. Evidence on the size of assets: Similar to the K&S approach, one can just look directly at the size of assets that people hold. Ask: are they holding as much wealth as the LCH says that they should be? Find that, in fact, people hold little wealth. For example, median financial wealth (that is, not counting house or pension or NPV of future Social Security) of families with head aged 45-54 was $4,131 in 1983. So it doesn't look like life cycle saving.

But there are some mitigating points here. First, most people have their retirement saving done for them in the form of SS and pensions. So maybe they are even saving too much in these forms, and don't want to save more -- maybe they would even save less, if they could. Also, people hold houses, which are very valuable. But in any case, assets seem so close to zero that it is hard to believe that they represent some optimum choice [note: it is possible that people set their pensions so that non-pension assets are zero...].

5. Campbell and Mankiw -- time series tests of the response of consumption to income. As we have seen above, there is no problem with consumption responding to changes in income. It may simply be that the change in income contains information about future changes in income. But suppose that we could look only at predictable (in advance) changes in income. PIH/LCH says that consumption should not change in response to these. This is what Cambell and Mankiw look at

\[ Y_t = \text{total disposable income}. \]

Suppose divided into two streams: to PIH people (group 2) and to people who consume current disposable income (group 1). $\lambda$ goes to group one.

\[ C_{1,t} = Y_{1,t} = \lambda Y_t \]
\[ Y_{2,t} = (1-\lambda) Y_t \]

in differences:

\[ \Delta C_{1,t} = \Delta Y_{1,t} = \lambda \Delta Y_t \]
While for the PIH groups
\[ C_{2,t} = Y_{2,t}^p = (1-\lambda)Y_t^p \]
\[ \Delta C_{2,t} = \mu + (1-\lambda)\epsilon_t, \]

where \( \mu \) is the trend growth rate (due to OLG and growth) and \( \epsilon \) is the innovation in \textit{permanent} income. (and is orthogonal to past...)

\[ \Delta C_t = \mu + \lambda \Delta Y_t + (1-\lambda)\epsilon_t. \]

Of course this can't be estimated directly, since change in \( Y \) is correlated with error in permanent income. (If there were never any shocks to permanent income, then \( \epsilon \) would be zero all the time, and we could estimate the equation directly).

The solution is to find instruments for Delta \( Y \) -- lagged things that predict changes in income.

The solution is to use twice-lagged income -- see my notes for 284 for further discussion (or see the article itself or Deaton's book).

They estimate \( \lambda \) to be around .5 -- so half of all consumption is done by rule-of-thumb (or liquidity constrained) consumers. (This does not mean that half of people are this way -- since it is probably the poorer people who are more likely to be liquidity constrained, more than half of actual people will be liquidity constrained if half of consumption is done by such people.).

Shea

(not the paper discussed in the Romer book)

Same framework, but examine the possibility that liquidity constraints are the problem.

Divide changes in income into pos and neg. It should be for Pos where consumption follows income -- so it should have a larger lambda.

Unfortunately, there are few drops in income, and fewer predictable (at least at the agg level)

More realistically: divide into above and below trend.

Table: perverse finding: consumption responds more to predictable changes in income when
income is going down than when it is going up.

Kubler Ross? (see quote).

Wilcox

changes in SS benefits announced at least 6 (and usually 8) weeks in advance. Under PIH, there should be no change in consumption in the months in which change actually occurs.

Coefficients say that there is a 1.4% change in consumption for a 10% increase in SS benefits. Note this does not say that consumption of the SS recipients goes up by 14% of their income change -- this is total consumption.

1986 SS payments 200 billion, PCE 2.8 trillion. So by point estimate, more than dollar for dollar spending.

Wilcox also finds that the increase is biggest in durable goods (and of these, biggest in autos) -- this seems like people are buying a car with their extra cash flow!

Many extensions of this work possible: look at consumption of old-people sensitive goods (big, slow cars, eg) or by location...

New Material on Consumption

More evidence for and against the PIH:

— monthly income is relatively constant throughout the year, but consumption expenditure rises significantly in December (in US, 21 percent.) This is actually good news for the PIH, even though consumption is non-smooth. The reason consumption rises is obviously because of preferences for holiday spending. The fact that consumption responds to this, rather than matching income, is thus good news for PIH.

Souleles (1999) Consumption rises when income tax refunds are received, even though this is predictable.

Parker (1999) Consumption rises when take home pay rises as a result of cessation of Social Security payments. This consumption is concentrated in durables and goods that can more easily be postponed.

— Browning and Collado (2001): in Spain, the majority of workers receive a double paycheck in June
and December - but there are some workers who do not receive such payments. (This depends on the worker's job - it is _not_ like a bonus that is uncertain.) Finding is that the seasonal pattern of expenditure is the same in two groups. This is a big win for PIH.

[Possible reconciliation: in the Spanish case, the cost of not smoothing would be large. In the US cases, the non-smoothness is small, so no big cost from just having consumption move with income.

One-time Payments:

- 1950, unanticipated payments to subset of US veterans holding National Service Life Insurance policies. MPC = .3 - .5. Similarly, reparations payments from Germany to certain Israelis 1957-58, MPC = .2 (these were very large payments, equal to about one year's income). Carroll points out that when these cases were originally analyzed, they were seen as victories for the Friedman PIH since the MPC was much less than one. But in modern view, they are failures, since the MPC is much higher than .05. Carroll's point is that the modern PIH is not the same as the original Friedman PIH.


Toward reconciling all (some?) of this stuff:

1. Liquidity constraints prevent borrowing

2. Many people would like to borrow (either because income is growing rapidly and/or because they have a high discount rate).

3. People don't hold zero assets because of a precautionary motive: income is stochastic and they may get a bad draw.

4. Optimal strategy (ala Deaton and Carroll) is to hold a buffer stock of assets -- say a few months worth of income.

[could do a little exercise to show this... or go over Deaton's version.]

5. How explain the aggregate capital stock? ==> rich people are not subject to the above model. They hold most of the wealth. They are not subject to any of the models that we know.

6. Maybe some room for LCH/PIH people in the middle
7. Maybe more folks would optimize if they had to
Problems

1. The Coefficient of Absolute Risk Aversion is defined as

\[
\frac{U''(C)}{U'(C)}
\]

and the Coefficient of Relative Risk Aversion is defined as

\[
\frac{U''(C)C}{U'(C)}
\]

where \(U'(C)\) is the first derivative and \(U''(C)\) is the second derivative of the utility function.

Consider the following two utility functions:

1. \(U(C) = \frac{1}{\alpha}e^{-\alpha C}\)
2. \(U(C) = C^{1-\sigma}/1-\sigma\)

In which case is the coefficient of absolute risk aversion invariant to the level of \(C\)? (this is called the “constant absolute risk aversion utility function”). In which case is the coefficient of relative risk aversion invariant to the level of \(C\)? (this is called the “constant relative risk aversion utility function”).

1.5) [core exam, 2006] A researcher is trying to determine the parameters of a woman’s utility function. He knows that she has CRRA utility, but he does not know the value of her coefficient of relative risk aversion, \(\sigma\), or her time discount rate, \(\theta\). The woman is infinitely lived. Time is continuous.

The researcher has presented the woman with different possible paths of consumption, asking which was preferred to which. The woman answered that she was indifferent among the following three paths of consumption:

\[
c(0) = 1, \quad g = .01 \quad c(0) = 2, \quad g = 0 \quad c(0) = 4, \quad g = -.0025
\]

where \(c(0)\) is initial consumption and \(g\) is the annual growth rate of consumption.

Based on this information, solve for the woman’s values of \(\sigma\) and \(\theta\). Note: solving this in general would require a computer. However, I will make things easier by telling you that \(\sigma\) is equal to either 2 or 3.
2. Consider a two period Fisher model in which utility in each period is given by the function

$$U(C) = C^{1-\sigma}/1-\sigma$$

Solve for the derivative of saving with respect to the interest rate: $dS/dr$ (this is just the negative of the derivative of first period consumption w.r.t. the interest rate). What condition on the values of $w_1$ and $c_1$ guarantees that $dS/dr$ is positive? Assuming that this condition does not hold true, how (informally) does the value of $\sigma$ affect whether $dS/dr$ is positive or negative.

3. An individual lives for two periods. In the first period she earns a wage of 100. In the second period she earns a wage of zero. She earns interest on her savings at some interest rate $r>0$. Her within-period utility function is

$$U(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$$

She has a discount rate of zero.

For what values of $\sigma$ will her first-period consumption be equal to 50? For what values of $\sigma$ will it be less than 50? For what values will it be greater than 50?

4. [midterm exam, 2005] A woman is born at time zero and will live forever. Time is continuous. She has initial assets of zero. She can borrow or lend at interest rate $r=0.04$. Her time discount rate is $\theta = 0.02$. Here wage at time zero, $w(0) = 1$. Here wage grows at a rate of 2% per year. She has CRRA utility, with a coefficient of relative risk aversion of 2. Solve for her consumption at time zero.

5. Consider the optimal consumption path of a person who will live exactly $T$ periods. In the first period, she has labor earnings of one unit. Subsequently, her wage grows at rate $g$, so that in the second period she earns $1+g$, in the third period $(1+g)^2$, etc. She can borrow or lend at an interest rate of zero, and she discounts future utility at rate zero. Her instantaneous utility function is of the CRRA form. She starts and ends life with zero assets.

Calculate saving in the first period of life (that is, first period income minus first period consumption). What is the effect of increasing $g$ on first period saving? Explain.
6. A man lives for 40 periods. He has constant absolute risk aversion utility:

\[ U(c_t) = \frac{-1}{\alpha} e^{-\alpha c_t} \]

He earns a wage of 100 in each period. The interest rate is \( r \) and the time discount rate is \( q \). He is born with zero assets and will die with zero assets, but he is able to borrow during his life (i.e. he is not liquidity constrained). There is no uncertainty.

A) Derive (or state) the first-order condition relating consumption in adjacent periods of his life.

B) Assume that \( \alpha = 1 \), \( q = 0.05 \), and \( r = 0 \). Derive his optimal first period consumption. \textbf{Note:} you will have to use the approximation that \( \ln(1+x) = x \), which holds true for values of \( x \) near zero.

7. A researcher has collected income and consumption data from a large population. Consumption in the population is determined by the permanent income hypothesis: \( C = \alpha Y^p \), where \( Y^p \) is permanent income and \( 0 < \alpha < 1 \). Permanent income in the population is given by \( Y^p_i = \rho_i \), where \( \rho \) is distributed normally, with variance \( \sigma^2_p \). The researcher does not observe permanent income, however. She only observes current income, which is related to permanent income by \( Y^c_i = Y^p_i + \varepsilon_i \). \( \varepsilon_i \) is transitory income. It is distributed normally, with variance \( \sigma^2_t \). There is zero covariance between permanent and transitory income.

The researcher estimates the "consumption function"

\[ C_i = \beta_0 + \beta_1 Y^c_i \]

What value will she get for \( \beta_1 \), the marginal propensity to consume? How will \( \beta_1 \) compare to \( \alpha \)? Under what circumstances will it be a good estimate, and under what circumstances a bad estimate?

8. [final exam, 2001] In response to the current economic slowdown, some economists have proposed a temporary reduction in sales taxes in order to stimulate consumption. The following question is inspired (loosely) by that proposal.

Consider the problem of optimal consumption in the presence of sales taxes. Let \( e_t \) be expenditure on consumption in period \( t \) (that is, what the household spends). Let \( \tau_t \) be the tax rate in period \( t \). The relation between consumption, expenditure, and taxes in a period is
\[ c_t = (1 - \tau_t) e_t \]

We assume that all tax collections are thrown away.

Consider a person who lives for two periods. He has wealth \( A_1 \) at the beginning of the first period. He earns no wage income. There is an interest rate \( r \) and time discount rate \( \theta \), where \( r = \theta > 0 \). The utility function is CRRA.

A) Suppose that tax rates in the two periods are equal, that is \( \tau_1 = \tau_2 = \tau > 0 \). Solve for optimal consumption and expenditure in each period.

B) For what values of \( \sigma \) will a decrease in the first period tax rate lead to an increase in first period expenditure?

C) For what values of \( \sigma \) will a decrease in the first period tax rate lead to an increase in first-period tax revenue?

9. [midterm exam, 2001] So far, we have been ignoring changes in family composition when we studied consumption. In fact, however, such changes – for example the addition of children to a household – have an important effect on consumption.

Consider the case of a household that lives for two periods. In the first period there are \( M_1 \) members of the household, and in the second period there are \( M_2 \) members. The interest rate and time discount rates are both zero. The present discounted value of lifetime wages (of all members of the household) is \( w \). Let \( c_t \) be total consumption of the household in period \( t \). We assume that consumption is split evenly among the members of the household. We will consider two different possible ways of modeling the way in which consumption affects utility. In each case, solve for the optimal level of total household consumption, \( c \), in both periods.

A) Suppose that consumption is split evenly among members of the household, and that total household utility in a period is equal to per-person utility multiplied by the number of people in the household. Per-person utility, in turn, is just given by the CRRA utility function. So total utility is:

\[ U_t = M_t \left( \frac{(c_t / M_t)^{1-\sigma}}{1-\sigma} \right) \]

B) Suppose that consumption is split evenly among members of the household, and that total household utility in a period is equal to average per-person utility. So the total utility function is:
Problems with Liquidity Constraints

10. Consider the following version of the Fisher model. Individuals live for two periods. They can borrow at a real interest rate of 100% (that is, if they borrow one dollar in period 1, they must repay two dollars in period 2). They can lend at a real interest rate of zero. Their preferences are given by:

\[ U_t = \left( \frac{c_t}{M_t} \right)^{-\sigma} \]

Find optimal first period consumption for the following three individuals:

Ms X: \( w_1 = 32 \) \( w_2 = 32 \)
Ms Y: \( w_1 = 0 \) \( w_2 = 64 \)
Ms Z: \( w_1 = 24 \) \( w_2 = 40 \)

11. (midterm, 2001) In our previous analysis of differential interest rates we assumed (realistically) that the interest rate on borrowing was higher than the interest rate for saving. Now we make the opposite assumption, which is not all that realistic.

Consider a person who lives for two periods. The interest rate for saving is 100%, so that one dollar saved in period one turns into two dollars in period 2. The interest rate on borrowing is zero.

A woman has labor income \( w_1 = 2 \) and \( w_2 = 3 \). She has log utility and a time discount rate of zero.

Draw the set of feasible consumption possibilities. Show how to solve optimal consumption in each period. It turns out that in figuring out optimal consumption, the very last step requires the use of a calculator. So you should solve up to this last step and then just indicate the calculation that you would make to figure out optimal consumption.

12. Mr. A and Mr. B have the same preferences (that is, the same instantaneous utility function and the same discount rate). Both are born at time zero and die (with certainty) at time T. Both face
the same interest rate. Each is born with zero assets and dies with zero assets. Both are also liquidity constrained: they are never allowed to have negative assets. They have different lifetime wage profiles. There is no uncertainty: both individuals know in advance their entire lifetime wage profiles.

It is observed that Mr. A's consumption grows at a constant (positive) rate over the course of his life, while Mr. B's consumption declines at a constant rate over the course of his life. The rate at which Mr. B's consumption declines is smaller than the rate at which Mr. A's consumption grows. However, the present discounted values of lifetime consumption for the two men are the same.

Which man has higher lifetime utility? Explain.

13. [core exam, 2004] A woman is born at time zero and will live forever. Time is continuous. She is born with zero assets. Her instantaneous utility function is of the CARA form:

\[ U(c) = \frac{1}{\alpha} e^{-\alpha c} \]

where \( \alpha = 1 \). She discounts the future at rate \( \theta = 0.10 \). She can save at interest rate zero. She cannot borrow.

Her labor income is exogenous. From time zero to time \( t=10 \), her labor income is two per period. From time \( t=10 \) to time \( t=20 \), her labor income is one per period. After time \( t=20 \), her labor income is two per period.

Solve for her path of consumption. What is her consumption immediately before and after \( t=10 \)? What is her consumption immediately before and after \( t=20 \)? If there are any jumps or inflections in her time path of consumption, calculate the exact point in time at which they take place.

Note: the First Order Condition for consumption with CARA utility is

\[ \dot{c} = \frac{1}{\alpha} (r - \theta) . \]

This is analogous to the condition that we look at in the more common case of CRRA utility, which is

\[ \frac{\dot{c}}{c} = \frac{1}{\sigma} (r - \theta) \]
14. (Final exam, 2001) This question was going to be on the Core exam, but I realized that it was way too hard. So I have broken it down into steps and given you instructions as you go along.

A woman has labor income that is constant at one. Time is continuous. She is liquidity constrained, so that she can never have negative assets. She is born with zero assets. From time zero to time s, she faces an interest rate of zero. Starting at time s, she will be able lend at interest rate $r>0$. Her time discount rate is $\theta$, where $r>\theta>0$. Her instantaneous utility function is logarithmic.

Sketch her optimal time path of consumption. Under what conditions will optimal consumption at time zero be equal to one?

A. As a first step, solve for the path of optimal consumption starting at time s. Solve for consumption at time s, $c(s)$, as a function of her assets at time s, $A(s)$. You should be able to find a closed-form expression for $c(s)$. Make a graph with $A(s)$ on the horizontal axis and $c(s)$ on the vertical axis.

B. Now consider what happens before time s. It turns out that there are several possible paths for consumption in the period before time s, depending on the values of s, r, and $\theta$. Sketch out these possible paths.

C. Based on your answer to part B, draw a graph with $A(s)$ on the horizontal axis and $c(s)$ on the vertical axis, showing how assets at time s are related to consumption at time s. Show (with words or arrows) how the different points on this curve are related to the different consumption paths in part B.

D. Putting together your answers to parts A and C, along with the usual condition that consumption cannot jump, will give the optimal values of consumption and assets at time s.

E. Holding r and $\theta$ constant, how will changing s affect the value of consumption at time zero? Specifically, under what conditions will consumption at time zero be equal to one?

Problems with Endogenous Interest Rates

15. Consider a world with two countries, which have equal sized populations. Income is exogenous and is identical in the two countries. There is no means of storing output, so income in a given period has to be consumed in that period. There is no population growth.

Both countries have CRRA utility of the form
\[ u(c) = \frac{c^{1-\sigma}}{1-\sigma}. \]

But the two countries have different values of \( \sigma \). Specifically, \( \sigma_1 > \sigma_2 \). In both countries, the time discount rate is zero.

Income per capita at time zero is equal to one. Following this it grows at constant rate g.

A. Suppose that there were no trade between the countries. What would the interest rate be in each country?

B. Suppose that starting at time zero, the two countries are able to trade with each other. Sketch out what the path of consumption per capita will look like in each country. Also sketch out the path of the world interest rate. Toward what level will the world interest rate asymptote? What will the asymptotic growth rates of consumption in the two countries be, and what will be the asymptotic ratio of consumption in the two countries? How will consumption at time zero in the two countries compare?

Note: doing this problem with fancy math is neither required nor recommended.

16. [final exam, 2001] An economy is composed of equal numbers of two types of people, purple and green. People life for two periods. Their lifetime utility functions are

Green: \[ U = \ln(c_1) + \ln(c_2) \]
Purple: \[ U = \ln(c_1) + \beta \ln(c_2) \]

The world lasts for two periods. In each period, each person receives labor income of one. There is no capital or other way to store output. Green and purple people can, however, borrow from or lend to each other.

Is the equilibrium interest rate positive or negative? How does the answer depend on the value of \( \beta \)? Why? You can answer the question with intuition or by grinding through the math.

17. [midterm exam, 2003] Consider a world with two countries, which have equal-sized
populations. Time is discrete. The world lasts for 101 periods, starting in period zero and ending after period 100. Labor income is equal to 1 unit per capita in both countries in all periods. Output cannot be stored, but can be traded between the two countries. The interest rate is endogenous.

Within each country, instantaneous utility is of the Constant Absolute Risk Aversion (CARA) form:

\[ U = -\left(\frac{1}{\alpha}\right)e^{\alpha c} \]

The two countries have identical values of \( \alpha \), specifically \( \alpha = 1 \). The two countries have different time discount rates, however. Specifically, \( \theta_1 = 0.01 \) and \( \theta_2 = -0.01 \) (this means that people in country 2 care more about the future than the present).

Solve for the path of the real interest rate and for the initial level of consumption in each country.

Note: you will have to use the approximation that \( \ln(1+x) = x \), which holds for values of \( x \) near zero.

18. [final exam, 2003] The world is composed of two countries with equal populations. In both countries, output is exogenous. There is no capital or other means to store consumption from period to period, but people can make loans between the countries. In both countries, people have log utility with time discount rate \( \theta>0 \).

In country 1, output is equal to \( y_h \) in even periods and \( y_l \) in odd periods, where \( y_h > y_l \). In country 2, output is equal to \( y_l \) in even periods and \( y_h \) in odd periods. The world begins in period zero (which is, of course, an even period).

Solve for the path of interest rates, and also find the path of consumption in each country.

19) [Core Exam, 2005]. Consider a world with two countries, which have equal sized populations. Time is continuous. Labor income per capita in each country is constant, exogenous, and equal to 1. There is no means of storing output. There is no population growth. People in both countries have the following instantaneous utility functions.
\[ u(c) = -e^{-c} \quad \text{if} \quad c \geq 0 \]

\[ = -\infty \quad \text{if} \quad c < 0 \]

Individuals in country i discount the future at rate \( \theta_i \) (i = 1, 2), where

\[ 0 < \theta_1 < \theta_2 \]

Starting at time zero, individuals in each country are allowed to borrow or lend at the market-clearing world interest rate.

Note: you do not have to write out any equations at all in order to get full credit for this problem, but you should carefully describe the principles that determine the answer you give. Because this problem is very difficult, I have broken it down into smaller steps.

A. Suppose that at some point in time, \( c > 0 \) in both countries. What must the interest rate be?

B. Suppose that the interest rate is the value that you derived in part A. What does the path of consumption (at that point in time) look like in each country?

C. Explain why based on the above, the interest rate cannot permanently have the value you derived in part A.

D. Suppose that at a point in time, consumption is zero in country 1 and positive in country 2. What must the interest rate be? What do the paths of consumption look like in each country?

E. Is it possible that the interest rate will have the value you derived in part D from time zero until infinity? Explain.

F. Suppose that at a point in time, consumption is zero in country 2 and positive in country 1. What must the interest rate be? What do the paths of consumption look like in each country?

G. Is it possible that the interest rate will have the value you derived in part F from time zero until time infinity? Explain.

H. Putting all of the above together, sketch out the possible time paths of consumption in each country as well as the world interest rate. If there are different possible cases, you should briefly describe these.

20. A man is born at time zero and will live forever. Time is discrete. The only thing he consumes
is wheat. His within-period utility function is $U = \ln(c)$, where $c$ is consumption of wheat. His time discount rate is $\theta > 0$. At the beginning of period zero, he has some quantity $A_0$ of wheat.

There are two things that he can do with any wheat that is left over at the end of a period. He can store it in his warehouse or he can plant it. If he stores it in his warehouse, then a fraction $\delta$ will decay before the beginning of the next period (where $0 < \delta < 1$). Alternatively, he can plant the wheat in the ground. Wheat planted at the end of period $t$ is harvested at the beginning of period $t+2$. One unit of wheat planted at the end of period $t$ yields $(1+\phi)$ units of wheat at the beginning of period $t+2$, where $\phi > 0$. Wheat planted at the end of period $t$ is not available at all in period $t+1$.

A. Solve for his path of consumption. Draw a picture showing what the path looks like. Carefully describe any interesting features of the consumption path, and explain why these interesting features are present. Also solve for his first period consumption (this expression is a bit messy. You don’t have to simplify it).

B. Now suppose that there are a large number of identical farmers. All start at time zero with identical quantities of wheat and face the same technology for production and storage described above. Solve for the path of interest rates on one-period loans of wheat. That is, what will be the interest rate on loans from period 0 to period 1; from period 1 to period 2; and so on.

Problems with Uncertainty

21. [midterm exam, 2003]

A) A woman has assets $A(0)$ at time zero. Time is continuous. She has no labor income. The interest rate and time discount rate are both zero. Her instantaneous utility function is $u(c) = \ln(c)$. She has a constant probability of death $\rho > 0$. Solve for her initial level of consumption, $c(0)$.

B) A woman has assets $A(0)$ at time zero. Time is continuous. She has no labor income. The interest rate and time discount rate are both zero. Her instantaneous utility function is $u(c) = \ln(c)$. From time zero to time 1, she has zero probability of death. Starting at time 1, she has a constant probability of death $\rho > 0$. Solve for her initial level of consumption $c(0)$.

C) A woman has assets $A(0)$ at time zero. Time is continuous. She has no labor income. The interest rate and time discount rate are both zero. Her instantaneous utility function is $u(c) = \ln(c)$. From time zero to time 1, she has zero probability of death. At time 1, she will have a constant probability of death. However, she will not find out that probability until time one.
Specifically, what she knows at time zero is that the probability of death starting at time 1 will be $\rho_1$ with probability 50% and $\rho_2$ with probability 50%. Solve for her initial level of consumption, $c(0)$ [Note: to solve this problem you have to deal with a nasty quadratic equation. You shouldn't try to solve this. Just derive it, then say what you would do with the solution if you could solve it.]

22. A man has a potential lifespan of 100 years. For the first 50 years, he will live with certainty. After 50 years, he will enter a battle, and there is a 50% chance that he will survive. If he survives the battle, then he will live for another 50 years and die at age 100.

At birth he has assets of $100. He does not earn any additional wage income. He cannot die in debt.

He has log utility: $u(c) = \ln(c)$. The interest rate and time discount rate are both zero.

Solve for his initial consumption, $c_0$.

23. (Midterm exam, 2001) A person may live for one, two, or three periods. At the end of period 1, after he has done his consumption for that period, there is a 50% chance that he will die, and a 50% chance that he will live into period 2 – he does not find out which happens until the event actually takes place. If he does live into period 2, then at the end of period 2, there is once again a 50% chance that he will die and a 50% chance that he will live into period 3 – once again, he does not find out which happens until the event actually takes place. His wealth at the beginning of period 1 is $A_1$. He earns no wage income. The interest rate and time discount rate are both zero. He has log utility. Solve for his optimal consumption (if he is alive) in each period.

24. Consider the problem of an old person trying to decide how much to consume as she approaches the end of her life. The interest rate is $r$, there is no time discounting, and she has no labor income. She has some initial amount of wealth, $A_0$. She faces a constant probability, $p$, of dying each year. Thus her probability of being alive in year $t$ is $(1-p)^t$. Of course she only gets utility if she is alive. Her utility function is

$$E(U) = \sum_{t=0}^{\infty} (1 - p)^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

A. What is the relationship between her coefficient of relative risk aversion ($\sigma$) and the expected size of the bequest that she will leave? Be sure to distinguish between the case where $p>r$ and the case where $p<r$. Explain.
B. Suppose that $\sigma = 1$, so that the woman has log utility. Solve for her assets if she is still alive at the beginning of period $t$, $A_t$, as a function of $p$, $r$, $t$, and $A_0$.

24.5 [final exam, 2005] A man is born at time zero. Time is continuous. He has a probability of dying, $\rho$, that is a function of his age. Specifically,

$$\rho(t) = .01t$$

To be clear, this means that the man has a 1% per year chance of dying when he is one year old; a 2% per year chance of dying when he is two years old (if he didn’t die already), and so on.

The man’s instantaneous utility is logarithmic. His pure time discount rate, $\theta$, is negative. In other words, ignoring the possibility of death, he cares more about the future than about the present. Specifically, $\theta = -.05$.

The man has wage that is constant and equal to one per period. He can save at an interest rate of zero. He cannot have negative assets. His initial assets are zero.

Sketch out the man’s paths of consumption and assets. You do not have to solve for the exact path or the exact value of $c(0)$. However, you should label as much of the path as you can. Specifically, you should point out any inflections, jumps, maxima or minima, etc. You should also give the dates of any of these points if you can, and if you can’t give the exact date, then you should say whatever you can (for example, “this jump takes place after date $x$ and before date $y$.“)

25. An individual lives for two periods. In each period she earns labor income of 1. Her utility function is

$$U = \ln(c_1) + \ln(c_2)$$

She can borrow or lend some financial asset that pays a real interest rate of zero. However, there is a 50% chance that between periods 1 and 2 all debts and financial wealth we be wiped out (there is no other way that she can save between periods other than the financial asset). That is, if she borrows, there is a 50% chance that she will not have to pay back the loan, and if she saves, there is 50% chance that all her saving will disappear. Solve for her optimal first period consumption.

26. An individual lives for two periods with certainty and for a third period with probability 50%. He finds out at the end of period one (after he has done his consumption) whether he will die at the end of period two or at the end of period three. He has earnings of one unit in the first period, and no labor income thereafter. His instantaneous utility function is

$$U(C_t) = C_t^{1-\sigma}/(1-\sigma)$$
He can borrow or lend at an interest rate of zero, and has a time discount rate of zero. He cannot die in debt.

A. Derive the equation that you would solve to get his optimal first period consumption. You don't need to solve it.

B. A government program is introduced that takes away $\tau$ from every worker in the first period of life, and pays $2\tau$ to all people who survive until the third period of life. What is the value of $\tau$ that maximizes expected utility? (You don't have to do algebra to get this answer -- you can just reason it out and explain your reasoning).

27. [midterm exam, 2002] A man has a maximum lifespan of three periods. In the first two periods, he is alive with probability one. At the beginning of the second period (before he has made his consumption decision), he finds out what his health status is. There are two possible values of health status, “healthy” and “unhealthy,” which each occur with a probability of 0.5. If he is healthy, then he will be alive in the third period with probability one. If he is unhealthy, he will be alive in the third period with probability 0.5. The information about whether he will be alive in the third period is not revealed until after second period consumption has taken place.

The interest rate and time discount rates are zero. He has labor income $W$ in the first period and zero in all subsequent periods. His instantaneous utility function is logarithmic.

Solve for his optimal first period consumption.

28. A person lives for two periods. In the first period she has income of 8 dollars. In the second period, she has income of either 0 or 8 dollars, each with probability 0.5. The interest rate is zero.

Her instantaneous utility function is:

$$U(c) = c - 0.05c^2$$

Her discount rate is zero.

There is a test that can tell the person what her second period income will be before she makes her first period consumption decision. If she does not take the test, then she will not know her second period income until after the first period is over. The test costs 2 dollars. Calculate whether she should take the test or not.
29. Consider a two-period model in which people work during both periods. The number of hours is fixed at 1 in each period. People start off with no assets, and die with no assets. They can borrow or lend at an interest rate of zero, and there is no time discounting. Their instantaneous utility function is \( U(C) = \ln(C) \).

In the first period, everyone's wage is $10. In the second period, half of the people will make $15, and half will make $5. But in the first period, people do not know to which group they will belong.

A) The government is considering cutting taxes by $1 per person in the first period, and increasing taxes in the second period to pay back the debt. Taxes in the second period can either be lump sum ($1 per person) or proportional (10% of each person's wage).

How would each of these programs affect people's utility and the national saving rate?

B) A test is invented that will tell people at the beginning of period 1 what their income will be in period 2. How would the availability of this test affect average utility of people in the population? How would it affect saving in the first period?

30. An individual faces the following problem. He lives for three periods. In the both the first and the second period, his income is 1 per period. In the third period, his income is either 1 or 2, each with a probability of .5. Further, he will find out at the end of the first period (that is, after he has done his first period consumption, but before he has done his second period consumption) what his income in period 3 will be. The interest rate and the person's time discount rate are both zero. His instantaneous utility function is \( U(C_t) = \ln(C_t) \). How would you solve for his consumption in the first period. (Don't worry about solving any nasty quadratic equations that arise: just show the equation that you would solve.)

31) [midterm exam 2004] A man will live for three periods. He has time discount rate of zero. His initial assets are \( A_0 \). He has no labor income. He has CRRA utility with a coefficient of relative risk aversion of 2.

The interest rate between periods 0 and 1 is zero. In period 1 (before he makes his consumption decision for that period) he will find out what the interest rate between periods 1 and 2 will be. Specifically, it will be zero with probability 50% and 3 with probability 50%.

Solve for his first period consumption. [Note: this final expression for \( c_0 \) is sort of ugly. You should at least derive the equation that implicitly gives \( c_0 \) as a function of \( A_0 \).]

32.[Core exam, 2003] A married couple is composed of a wife and a husband. Time is continuous.
The wife lives forever. The husband is alive at time zero, and has a constant probability of death \( \rho > 0 \).

All consumption decisions in the house are made by the wife. Her instantaneous utility function is

\[
U = \begin{cases} 
\ln(c_w) & \text{If the husband is alive} \\
\ln(c_w) & \text{If the husband is not alive}
\end{cases}
\]

where \( c_w \) is the consumption of the wife and \( c_h \) is the consumption of the husband.

The wife discounts future utility at rate \( \theta \). The interest rate is \( r \). Assume \( r = \theta > 0 \).

The household has assets at time zero \( A(0) > 0 \). In addition, while the husband is alive, the household receives labor income at a rate of one unit per period. Note that when the husband dies, all of the household's assets remain in possession of the wife.

Your job is to sketch the possible time paths of the wife's consumption and of household assets, that is \( c_w(t) \) and \( A(t) \), depending on the value of \( A(0) \) and on how long the husband lives. Specifically,

A) For what value of \( A(0) \) will the path of assets be flat? Call this value \( A^* \).

B) Consider the case where \( A(0) > A^* \). Sketch the possible paths of assets and wife's consumption. Specifically, show what the paths of consumption and assets look like before and after the husband's death. Show what the paths look like for different possible dates of the husband's death. Note: your answers to part B may be mostly or entirely in terms of words and pictures, rather than equations.

Problems with Nonlinear Budget Sets

34. A woman lives for two periods. In the first period she has income of 3. In the second period, she will have income of zero. The interest rate and the time discount rate are both zero. She cannot borrow. Her utility function is \( U(c_t) = \ln(c_t) \).

There is a government welfare program that provides a consumption floor of \( c_{\text{min}} \) in the second period. In other words: if she does not have enough money left over to afford to consume \( c_{\text{min}} \), then the government will give her enough money so that she can afford to consume it.

A) Obviously, for high enough values of \( c_{\text{min}} \), the consumer will decide to consume all of her wages in the first period and go on welfare in the second period. For low enough values of \( c_{\text{min}} \) she
will choose consumption as if the welfare program did not exist. Calculate the critical value of $c_{\text{min}}$ for which she is indifferent between these two strategies.

B) Now suppose that there is only a 50% chance that she will be alive in the second period. Calculate the critical value of $c_{\text{min}}$ at which she will be indifferent between the two strategies discussed in part A.

35. People live for two periods. In the first period they have income of 1, in the second period they have income of zero. The interest rate and the time discount rate are both zero. The utility function is $U(c_t) = \ln(c_t)$.

There is a 50% probability that there will be a government welfare program in the second period of people's lives. If there is such a program, it will provide a consumption floor of $c_{\text{min}}$. In other words: if they do not have enough money left over to afford to consume $c_{\text{min}}$, then the government will give them enough money so that she can afford to consume it.

Solve for the critical level of $c_{\text{min}}$ such that, if $c_{\text{min}}$ is below this level, people's choice of first period consumption will not be affected by the existence of the program.

The algebra on this question may get a little tedious. If you can't solve it, then show clearly what equation your would solve to get the answer.

36. An individual has wealth $A_0$ at time zero. She will live infinitely, and will not receive any income. The interest rate is zero. She is unable to borrow. Her instantaneous utility function is $U(c) = \ln(c)$. She has a discount rate of $\theta > 0$. Time is continuous.

There exists a government welfare program that works in the following manner. If a person has any wealth at all, she will receive nothing. If her wealth is equal to zero, then she will receive $c_{\text{min}}$.

Figure out her optimal consumption strategy. Solve for her initial consumption, $c_0$.

Hint #1: Try thinking about the case where $\theta = 0$.
Hint #2: Do not try thinking about this problem in discrete time. Doing so will make you insane.

37. [Core exam, 2001] A person is born at time zero and will live forever. He is born with assets of 10. His labor income is one per period. His time discount rate is 5%. His instantaneous utility function is of the CARA form:

$$U(c) = -e^{-c}$$
There are two interest rates in the economy. For borrowing, the interest rate is 5%. For saving, the interest rate is zero.

Solve for his optimal path of consumption. What is the consumption at time zero? Draw a picture of the time path of consumption, showing any inflection points, etc.

Note: Solving this problem in continuous time requires optimization techniques that we did not cover in my class. It can be more easily solved in discrete time. However, to do so, you have to use the approximations that $\ln(1+x) \approx x$ and $\ln(1/(1+x)) \approx -x$, both of which hold for $x$ near zero.

38) [Core Exam, 2005] An individual lives for one period with certainty and may live into a second period with probability $\rho$, where $0 < \rho < 1$. He knows the value of $\rho$. He does not find out whether he lives in the second period until after he has done his first period consumption. He has labor income $w=2$ in the first period of life, and no labor income in the second period of life.

His first period utility is $\ln(c_1)$

His second period utility is $\ln(c_2)$ if $c_2 \geq c_1$

$\ln(c_2) - 1$ if $c_2 < c_1$

The time discount rate and interest rate are both zero.

Make a graph showing his optimal first period consumption as a function of $\rho$. If there are any notable jumps or kinks in this function, you should write out the implicit equation for the value of $\rho$ at which they take place. You do not have to solve explicitly for the value(s) of $\rho$.

Problems with nonstandard utility functions

39. [midterm exam 2002] A woman lives for two periods. Her wage income is $W$ in the first period and zero in the second period. The interest rate and time discount rates are zero.

She has a habit-formation utility function of the following form:

$u_t = \ln(c_t - \gamma z_t)$ \quad $\gamma < 1$,

where $z_t$ is the habit stock period $t$. We assume that habit stock in a period is equal to last period’s consumption.
We further assume that habit stock in the first period of life is zero.

Solve for optimal consumption in the first and second periods of life.

40. [final exam, 2003] A person will be alive for three periods, labeled 1, 2, and 3. The interest rate is zero. He has initial assets $A_1$. His utility function in period $t$ (for $t = 1, 2$) is of the "hyperbolic" form,

$$U_t = \ln (c_t) + \beta \sum_{s=t+1}^{3} \ln (c_s)$$

where $\beta < 1$.

Solve for his path of consumption in periods 1, 2, and 3 under the following two scenarios:

A) [10 points] At time 1, he can decide his consumption in the current period as well as all future periods – that is, he can pre-commit himself to a lifetime path of consumption.

B) [10 points] At time 1, he can decide his consumption in the current period, but cannot pre-commit his future consumption path.

41. [midterm exam 2004] A woman is born at time zero and will live for exactly 9 periods, labeled $t=0, 1, ...8$. She has initial assets $A_0$. The interest rate and time discount rate are both zero. She has "hyperbolic" preferences of the form

$$V_t = \ln (c_t) + \beta \sum_{s=t+1}^{T-1} \ln (c_s)$$

where $\beta = \frac{1}{2}$.

A) Suppose that at time 0 she chooses her entire consumption path in periods 0-8. Further, there will be no subsequent revisions in her plans – that is, her hands will be tied in future periods. Solve for consumption in each period.

B) Suppose that in each period the person makes a consumption plan as if she were able to bind the hands of her future selves, and then does her consumption in that period according to the plan. So,
for example, in period 0 she will do the consumption that found in part A. But further suppose that in each future period, she is able to re-make her plans (and she again assumes, incorrectly, that her future selves will stick to that plan.).

Solve for her consumption in each period 0-8. You don't have to literally solve for each period, but rather show the general rule that will generate these values.

42. [midterm exam, 2003] A woman is born at time zero and will live until time T. Time is continuous. There is no uncertainty. She is born with zero assets and will die with zero assets. Her wage per unit of time is constant and equal to w. The interest rate is zero.

The woman has an unusual instantaneous utility function. It is

\[
u(c(t)) = \begin{cases} 
\alpha c(t) & \text{if } c(t) \leq c \\
\alpha c + \beta (c(t) - c) & \text{if } c(t) > c 
\end{cases}
\]

where \( \alpha > \beta \).

Her lifetime utility is given by

\[
V = \int_0^T e^{-\theta t} u(c(t)) \, dt
\]

where \( \theta > 0 \).

In the four parts of the question below, I ask you to solve for the optimal path of consumption under different assumptions about the values of the parameters. In each case, you don't have to give me exact values, although you may if you want to. Mostly, I want you to show me a picture of what the path looks like and briefly explain its key features.

A) Suppose that \( w = c \) and \( e^{-\theta T} > \beta \). Solve for the optimal path of consumption. Also, explain why the assumption that \( e^{-\theta T} > \beta \) is important.

B) Suppose that \( w > c \) and that \( e^{-\theta T} > \beta \). Solve for the optimal path of consumption.

C) Suppose that \( w < c \) and that \( e^{-\theta T} > \beta \). Solve for the optimal path of consumption.
D) Suppose that \( w = c \) and that \( \alpha e^{-\theta T} < \beta. \) Solve for the optimal path of consumption.

**Problems not elsewhere classified**

43) [midterm exam 2004] Time is continuous. A man is born at time zero and will live forever. He has initial assets \( A(0) = 100. \) He has no labor income. Assume that \( r=0 \) and \( \theta=0.5. \)

There are two things on which the man can spend his money. He can buy consumption goods (denoted \( c \)) or he can give his money to charity (denoted \( g \)). His instantaneous utility is given by

\[ U(c, g) = \ln(c) + g \]

Note that utility from gifts to charity has non-decreasing marginal utility, because it is assumed that one individual's contributions have an infinitesimal effect on the world's suffering.

Solve for his optimal paths of consumption and charitable contributions.