Math Camp Notes: Basic Proof Techniques

Basic notation:

- $A \Rightarrow B$. A implies B.
- $A \Leftarrow B$. B implies A. Note that $A \Rightarrow B$ does not imply $B \Rightarrow A$. Example: If (A) a person eats two hot dogs, she also (B) eats one hot dog. However, if (B) a person eats one hot dog, that does not implie that she also (A) eats two hot dogs.
- $A \Leftrightarrow B$. A implies B and B implies A. Another way of saying this is that A holds if and only if (iff) B holds, or that A is equivalent to B.
- $\neg A$. Not A, or the negation of A. Example: If A is the event that $x \leq 10$, then $\neg A$ is the event that $x > 10$.

We seek for ways to prove that $A \Rightarrow B$.

Direct Proofs

Deductive Reasoning

A direct proof by deductive reasoning is a sequence of accepted axioms or theorems such that $A_0 \Rightarrow A_1 \Rightarrow A_2 \Rightarrow \ldots \Rightarrow A_{n-1} \Rightarrow A_n$, where $A = A_0$ and $B = A_n$. The difficulty is finding a sequence of theorems or axioms to fill the gaps.

Example:

Prove the number three is an odd number.

Proof: Any number q is odd if there exists an integer m such that $q = 2m + 1$. Let $m = 1$. Then three is an odd number.■

Contrapositive

A contrapositive proof is just a direct proof of the negation. If we want to prove that $A \Rightarrow B$, then we can prove that $\neg B \Rightarrow \neg A$. For example, if ($A$) all people with driver's licenses are (B) at least 16 years old, then if you are not ($\neg B$) 16 years old, then you do not ($\neg A$) have a driver's license. Or at least we hope.

Example:

Prove that is $xy > 9$, then either $x > 3$ or $y > 3$.

Proof: Suppose that both $x \leq 3$ and $y \leq 3$. Then $xy \leq 9$.■

Indirect Proofs

Contradiction

Suppose that we are trying to prove a proposition A, and we cannot prove it directly. However, we can show that all other alternatives to A are impossible. Then we have indirectly proved that A must be true. Therefore, the we can prove $A \Rightarrow B$ by first assuming that $A \not\Rightarrow B$ and finding a contradiction.

In other words, we start off by assuming that A is true but B is not. If this leads to a contradiction, then either B was actually true all along, or A was actually false. But since we assume A is true, then it must be tthat B is true, and we have a proof by contradiction.

Example: Prove that $\sqrt{2}$ is an irrational number.

Proof: Suppose not. Then $\sqrt{2}$ is a rational number, so it can be expressed in the form $\frac{p}{q}$, where p and q are integers which are not both even. This implies that

$$2 = \frac{p^2}{q^2} \Rightarrow 2q^2 = p^2,$$

which implies that p^2 is even, which in turn implies that q^2 is not even. The fact that p^2 is even also implies that p is even, so there exists a integer m such that $2m = p$. This implies

$$4m^2 = p^2 = 2q^2 \Rightarrow q^2 = 2m^2,$$

which means that q is even, a contradiction.■

Induction

Induction can only be used for propositions about integers or indexed by integers. Consider a list of statements indexed by the integers. Call the first statement $P(1)$, the second $P(2)$, and the nth statement $P(n)$. If we can prove the following two statements about the sequence, then every statement in the entire sequence must be true:

1. $P(1)$ is true.
2. If $P(k)$ is true, then $P(k+1)$ is true.

Induction works because by 1., $P(1)$ is true. By 2., $P(2)$ is true since $P(1)$ is true. Then $P(3)$ is true by 2. again, and so is $P(4)$ and $P(5)$ and $P(6)$, until we show that all the P's are true. Notice that the number of propositions need not be finite.

Example:

Prove that the sum of the first n natural numbers is $\frac{1}{2}n(n+1)$.

Proof: Let $n = 1$. Then $\frac{1}{2} \cdot 1(1+1) = \sum_{j=1}^{1} j = 1$. Now let $n = k$, and assume that $\sum_{j=1}^{k} j = \frac{1}{2}k(k+1)$. We add $k+1$ to both sides to get

$$\sum_{j=1}^{k+1} j = \frac{1}{2}k(k+1) + k + 1 = \left(\frac{1}{2}k + 1\right)(k+1) = \frac{1}{2}(k+1)((k+1)+1).■$$

Proof Notation

It is common to use mathematical symbols for words while writing proofs in order to write faster. The following are commonly used symbols:

$\forall$	For all, for any
$\exists$	There exists
$\in$	Is contained in, includes as an element
$\ni$	Such that, is contained in (other way)
$\subset$	Is a subset of

Homework

Prove the following by direct proof.

1. $n(n+1)$ is an even number.
2. The sum of the first n natural numbers is $\frac{1}{2}n(n+1)$.
3. If $6x + 9y = 101$, then either x or y is not an integer.

Prove the following by contrapositive.

1. $n(n+1)$ is an even number.
2. If $x+y>100$, then either $x>50$ or $y>50$.

Prove the following by contradiction.

1. $n(n+1)$ is an even number.
2. $\sqrt{3}$ is an irrational number.
3. There are infinitely many prime numbers.

Prove the following by induction.

1. $n(n+1)$ is an even number.
2. $2n \leq 2^n$.
3. $\sum_{i=1}^{n} i^2 = \frac{1}{6} n\,(n+1)\,(2n+1)$.
4. The sum of the first n odd integers is n^2 (This is the first known proof by mathematical induction, attributed to Francesco Maurolico. Just in case you were interested.)

Find the error in the following argument, supposedly by induction:

If there is only one horse, then all the horses are of the same color. Now suppose that within any set of n horses, they are all of the same color. Now look at any set of $n+1$ horses. Number them $1, 2, 3, \ldots, n, n+1$. Consider the sets $\{1, 2, 3, \ldots, n\}$ and $\{2, 3, 4, \ldots, n+1\}$. Each set is a set of n horses, therefore they are all of the same color. But these sets overlap, therefore all horses are the same color.

In first semester micro you will be introduced to preference relations. We say that $x \succeq y$, (read "x is weakly preferred to y") if x is at least as good as y to the agent. From this, we can derive two important relations:

- The strict preference relation, $\succ$, defined by $x \succ y \Leftrightarrow x \succeq y$ but not $y \succeq x$. The strict preference relation is read "x is strictly preferred to y".
- The indifference relation, $\sim$, defined by $x \sim y \Leftrightarrow x \succeq y$ and $y \succeq x$. The indifference relation is read "x is indifferent to y".

We say that a preference relation is rational if:

- $\forall\, x, y$, either $x \succeq y$ or $y \succeq x$.
- $\forall\, x, y, z$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.

Prove the following two statements given that preferences are rational:

1. If $x \succ y$ and $y \succ z$, then $x \succ z$.
2. If $x \sim y$ and $y \sim z$, then $x \sim z$.