Math Camp Notes: Difference Equations

A **difference equation** is an equation which evolves over discrete time intervals. For example, a difference equation would be a function which would tell us what the value of some variable y is for any given time t. We generally have some starting initial point for t (usually $t = 0$), and we may or may not have a terminal point for t. Consider the function

$$y_t = b^t y_0,$$

where $b \in \mathbb{R}$. This is a difference equation. We can plug whatever we want in for t and get the value of the function at that point. For example, if $b = 0.5$ and $y_0 = 1$, then we have the following evolution of y over time:

$$y_0 = 1,\ y_1 = 0.5,\ y_2 = .25, \ldots, y_n = \frac{1}{2^n}$$

Sometimes we are not given an explicit solution for y_t, but are given a rule of how y_t evolves over one period. For example,

$$y_{t+1} = (1 + b)y_t.$$

Solving Difference Equations

Say we have some intial condition y_0, and consider the difference equation

$$y_{t+1} = ay_t.$$

The equation can be solved iteratively by simply starting with $t = 0$, calculating y_1, then using that solution to calculate y_2, and so forth. For example:

$$y_0 = y_0$$

$$y_1 = ay_0$$

$$y_2 = ay_1 = a \cdot ay_0 = a^2 y_0$$

$$y_3 = ay_3 = a \cdot a^2 y_0 = a^3 y_0$$

$$\vdots$$

We can look at the pattern and infer that a solution to the difference equation will be of the form:

$$y_t = a^t y_0.$$

Relationship Between Continuous and Discrete Time

Subtract y_t from both sides of the equation

$$y_{t+1} = (1 + b)y_t$$

to get

$$\Delta y_t = y_{t+1} - y_t = by_t$$

Notice that this looks a lot like the differential equation

$$\dot{y} = by.$$

In fact, if instead of time moving in discrete units, time moved continuously, then the difference equation above would be the same as the differential equation. To see this, consider the difference equation governing the growth of an asset given a constant interest rate, and is compunded once per period:

$$A_{t+1} = A_t(1 + r).$$

Now consider the same equation, except time now is measured in half-increments. In other words, the interest is compounded twice per period, but at the same annual rate r:

$$A_{t+1} = A_t\left(1+\frac{r}{2}\right)^2$$

Now consider the case where interest is compounded n times per period:

$$A_{t+1} = A_t\left(1+\frac{r}{n}\right)^n.$$

The limit of this function as $n \to \infty$ is

$$A_{t+1} = A_t e^r,$$

Solving this difference equation we have

$$A_t = A_0 e^{rt}$$

which is the solution to the differential equation $\dot{A} = rA$. Therefore, the equation $A_{t+1} = (1+b)A_t$ is the discrete time counterpart of the differential equation $\dot{A} = rA$.

Properties of the Solution $y_t = a^t y_0$

We have solved the differential equation $y_{t+1} = ay_t$. What happens to y_t as $t \to \infty$? There are seven cases, with subcases under them:

Case 1: $a > 1$ Here we can see that as $t \to \infty$, $a^t \to \infty$. Therefore, the system diverges to $+\infty$ or $-\infty$, depending on whether y_0 is greater or less than 0.

Case 2: $a = 1$ If $a = 1$, then the system will stay at its intial condition y_0, no matter what y_0 is.

Case 3: $a \in (0, 1)$ In this case, as $t \to \infty$, $a^t \to 0$. Therefore, $y_t \to 0$ as $t \to 0$. Also, the sequence of y_ts will be strictly monotonically decreasing if $y_0 > 0$, and monotonically increasing if $y_0 < 0$.

Case 4: $a = 0$ The system immediately jumps to $y_1 = 0$ no matter where y_0 is, and then $y_t = 0\ \forall\ t$ thereafter.

Case 5: $a \in (-1, 0)$ In this case, as $t \to \infty$, $a^t \to 0$. Therefore, $y_t \to 0$ as $t \to 0$. However, the sequence of y_ts will oscillate between positive and negative values as $t \to \infty$.

Case 6: $a = -1$ If $a = -1$, then $y_t = y_0$ for $t \in \{\text{even integers}\}$ $y_t = -y_0$ for $t \in \{\text{odd integers}\}$.

Case 7: $a < -1$ As $t \to \infty$, the subsequence $a^{t_e} \to \infty$ and $a^{t_o} \to -\infty$, where t_e denotes even integers and t_o denotes odd integers. Therefore, the system oscillates between positive and negative values, with the magnitude of oscillate growing to infinity over time.

Subcase to all 7 cases: $y_0 = 0$ In the case $y_0 = 0$, we have that $y_t = 0\ \forall\ t$.

Steady States

The condition for a steady state for differential equations was that $\dot{y} = 0$. For difference equations, the condition is similar:

$$\Delta y_t = 0 \Rightarrow y_{t+1} = y_t.$$

Intuitively, if we can find a value of y such which will be the same next period, then that level of y must be a steady state. Therefore, in our difference equation we plug in y_t for y_{t+1} and solve:

$$y_t = ay_t \Rightarrow y_t(1-a) = 0 \Rightarrow y_t = 0$$

Therefore, the steady state solution to our differential equation is $y = 0$. This makes sense, because we already saw during our analysis of cases that $y_t = 0\ \forall\ t$ when $y_0 = 0$.

Stability of Steady States

Looking at our seven cases for the asymptotics of $y_{t+1} = ay_t$, we see that we converge to the steady state $y = 0$ whenever $|a| < 1$. We can generalize this condition by noticing that $a = \frac{\partial y_{t+1}}{\partial y_t}$. Therefore, the system is

$$\text{unstable if } \left| \left. \frac{\partial y_{t+1}}{\partial y_t} \right|_{y_{ss}} \right| > 1$$

$$\text{stable if } \left| \left. \frac{\partial y_{t+1}}{\partial y_t} \right|_{y_{ss}} \right| < 1$$

This is very similar to the conditions for stability of differential equations.

Nonlinear example

Consider the following evolution of captial in an economy:

$$k_{t+1} = k_t - (\delta + n)k_t + sf(k_t)$$

This says that the level of capital per capita next year is equal to the level of capital this year, minus depriciated capital and capital dilution from population growth, plus the difference between output and consumption. We seek to anaylze the dynamics of capital in this economy. Assume that $f(k_t)$ is a concave function, and $f(0) = 0$, $f'(0) = \infty$, and $\lim_{n\to\infty} f'(n) = 0$. Also assume that $\delta + n < 1$, $\delta, n, s \in (0, 1)$.

First we wish to find a steady state level of capital per capita. We set $k_{t+1} = k_t$ to get

$$k_{t+1} = k_t - (\delta + n)k_t + sf(k_t) \Rightarrow$$

$$(\delta + n)k^* = sf(k^*).$$

This will pin down some level of capital k^*. Notice that $k^* = 0$ is a steady state, as will some $k^* > 0$ by the concavity of $f(k)$.

Checking for stability, we take the derivative of the function k_{t+1} with respect to k_t to get

$$\frac{\partial k_{t+1}}{\partial k_t} = 1 - (\delta + n) + sf'(k_t)$$

Therefore, we have that the system is stable if $f'(k^*) < \frac{\delta+n}{s}$. For the steady state $k^* = 0$, we have that $f'(0) = \infty$, and therefore the system is unstable in this case. However, by the concavity of $f(k_t)$ the steady state $k^* > 0$ must be stable since $\frac{\delta+n}{s}k^* = f(k^*)$. Therefore we have one stable steady state and one unstable steady state.

Homework

Solve the difference equation $y_{t+1} = by_t^2$. Find the steady states. Graph the function with respect to time. Draw the phase diagrams. Determine stability of the steady states. (Hint: there will be several cases.) What about $by_{t+1} = ay_t^2 + c$? (This is complicated, so think about it graphically for starters. Don't worry if you can't finish the whole problem with all the cases.)