Math Camp Notes: Differential Equations

A **differential equation** is an equation which involves an unknown function $f(x)$ and at least one of its derivatives. Let $y = f(x)$. Then we denote $f'(x^*)$ as $\frac{df}{dx}(x^*)$ or as $\dot{y}$. The purpose of this equation is not to solve for the variable x, but rather to solve for the function $f(x)$.

Types of Differential Equations

1. **1st order -** Equations which involve the first derivative $f'(x)$ of the function but no higher derivatives. For example:

   $$\dot{y} = ky$$

   is a first order difference equation since it only involves one derivative of $f(x)$.

2. **Nth order -** Equations which involve the nth order derivatives $f^{(n)}(x)$ of the function. For example:

   $$\ddot{y} = ky$$

   is a second order differential equation since it involves two derivatives of $f(x)$. Also

   $$y^{(n)} = ky$$

   is an nth order differential equation.

3. **Ordinary -** An ordinary differential equation is a differential equation with only one argument. For example, the differential equations mentioned thus far have all been ordinary. However, the equation

   $$y^{(n)} = ky(x) \cdot t$$

   is not ordinary, while

   $$y^{(n)} = ky \cdot x$$

   is ordinary (the only variable is x).

4. **Linear -** A differential equation is linear if it can be written in the form

   $$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y = a_0(x) \Rightarrow \sum_{i=1}^{n} a_i(x)y^{(i)} = 0.$$

5. **Autonomous -** An autonomous differential equation is one where the only occasion when x enters the equation is through y. For example

   $$\dot{y} = ky$$

   is autonomous while

   $$\dot{y} = ky \cdot x$$

   is not.

Solving Differential Equations

Differential equations are generally difficult to solve. Therefore, in this section of the course we will examine only first order linear difference equations:

$$\dot{y} + p(x)y = q(x).$$

Special Cases

Notice that if $p(x) = 0$, then we have a simple integration problem we all know how to solve

$$\dot{y} = q(x) \Rightarrow y = Q(x) + C,$$

where $Q(x)$ is the antiderivative of $q(x)$.

Now let $q(x) = 0$, and $p(x) = -k \in \mathbb{R}$. Then we must solve the equation

$$\dot{y} = ky \Rightarrow \frac{dy}{dx} = ky(x).$$

Rewrite the equation by dividing both sides by $y(x)$ and multiplying both sides by dx (We are assuming that $y(x) \neq 0$, and we will soon see that this is the case). Then we have

$$\frac{\dot{y}}{y} = kdx.$$

Integrating both sides we have

$$\ln(y) = kx + C \Rightarrow y = e^{kx+C} = \gamma e^{kx},$$

where $\gamma = e^C$.

General Case

In the general case of

$$\dot{y} + p(x)y = q(x),\ p(x) \neq 0,\ q(x) \neq 0$$

is solved in the following manner. Define a function $H(x)$ such that

$$H(x) = \int p(x)dx.$$

Now multiply the above differential equation to get

$$\dot{y}e^{H(x)} + p(x)ye^{H(x)} = q(x)e^{H(x)}$$

Notice that by the chain rule, that

$$\frac{d}{dx}\left(ye^{H(x)}\right) = \dot{y}e^{H(x)} + p(x)ye^{H(x)}$$

which is equal to the left hand side of the differential equation. Therefore

$$\frac{d}{dx}\left(ye^{H(x)}\right) = q(x)e^{H(x)}.$$

We now integrate both sides to get

$$ye^{H(x)} = \int q(x)e^{H(x)}dx + C,$$

and then multiply through by $e^{-H(x)}$ to get he general form of the solution:

$$y = e^{-H(x)}\left\{\int q(x)e^{H(x)}dx + C\right\}.$$

Example:

Find the solution of the equation $\dot{y} = ay + b$.

Solution: In this case $p(x) = -a$ and $q(x) = b$. Therefore

$$H(x) = \int -adx = -ax + C.$$

Plugging this into our general solution equation, we have that

$$y = e^{-ax+C}\left\{\int be^{-ax+C}dx + D\right\}$$

$$y = e^{ax-C}\left\{-\frac{b}{a}e^{-ax+C} + D\right\}$$

$$y = -\frac{b}{a} + De^{ax+C}$$

$$y = -\frac{b}{a} + \alpha e^{ax},$$

where $\alpha = De^C$.

Additional Conditions

Sometimes in a problem we are given an intial condition or a terminal condition. Sometimes we can use these conditions to help us find what the unknown scalars are in our solutions.

Example:

Let $y(0) = y_0$. Solve for the general solution to the problem $\dot{y} = ky$.

Solution: We already know the general solution to this problem is

$$y = \gamma e^{kx}.$$

We know that when $x = 0$, $y = y_0$. Plugging these values into the equation we have

$$y_0 = \gamma.$$

Therefore, the general solution to the equation is $y = y_0 e^{kx}$.

Example:

Let $y(T) = y_T$. Solve for the general solution to the problem $\dot{y} = ky$.

Solution: We already know the general solution to this problem is

$$y = \gamma e^{kx}.$$

We know that when $x = T$, $y = y_T$. Plugging these values into the equation we have

$$y_T = \gamma e^{kT} \Rightarrow \gamma = y_T e^{-kT}$$

Therefore, the general solution to the equation is $y = y_T e^{-kT} e^{kt} = y_T e^{k(t-T)}$.

Finding Steady States

A steady state for a differential equation is a solution where the value of y does not change over time. For example, consider an economy with capital and depriciation. For a given level of investment, we are either spending more that the depriciation of the capital stock, less that the value of the depriciation of the captial stock, or just equal. If we are equal, then we will have the same amount of capital next period as we do this period. Then if the level of investment is the same next year as it is this year, we will just cover depriciation with nothing left over. The capital stock will remain the same as this year in the third year, and so on. This is called the steady state level of capital.

Now let us consider the differential equation $\dot{y} = ay$. In order for the level of y to be the same this year and last year, we must have that y does not change, or $\dot{y} = 0$. Therefore, the only value of y for which this can happen is $y = 0$, and so $y = 0$ is a steady state to the equation.

Example:

Find the steady state for the equation $\dot{y} = b + ay$.

Solution: Let $\dot{y} = 0$. Then $ay = -b$, and the steady state value of the solution is $y = -\frac{b}{a}$.

Phase Diagrams

A **phase diagram** of a differential equation is a graph of the differential equation. Usually we have the level of the value of the function y on the horizontal axis, and the change in y on the vertical axis. Another important component of a phase diagram are arrows pointing the direction of the movement of y over time. For example, let us consider the simple differential equation

$$\dot{y} = ay.$$

Here there are two cases:

Case one: $a > 0$**.** In this case, the curve is a linear function sloping upward. What separates the phase diagram from a normal graph, however, is that we can draw arrows indicating the movement of y over time. Notice that if $a > 0$, then when $y > 0$, $\dot{y} > 0$. This implies that y is growing over time. If $y < 0$, then $\dot{y} < 0$, and y is shrinking over time. Therefore, we can see that if the equation starts at any point other than $y_0 = 0$, the system will diverge to negative infinity or positive infinity. Another was to see this is to take the solution to the equation, $y = y_0 e^{ax}$, and let $x \to \infty$. If $y_0 > 0$, then $y \to \infty$, and if $y_0 < 0$, then $y \to -\infty$.

Case two: $a < 0$**.** In this case, the curve is a linear function sloping downward. Notice that if $a > 0$, then when $y > 0$, $\dot{y} < 0$. This implies that y is shrinking over time. If $y < 0$, then $\dot{y} > 0$, and y is growing over time. Therefore, we can see that if the equation starts at any point, it will eventually converge to $y = 0$. Another was to see this is to take the solution to the equation, $y = y_0 e^{-ax}$, and let $x \to \infty$. If $y_0 > 0$, then $y \to 0$, and if $y_0 < 0$, then $y \to 0$ also.

Stability

We call a steady state x in the domain **stable** if $\exists\ r > 0$ and $B(x, r) \subset$ domain such that if we have as an initial point any $y \in B(x, r)$, the system will converge to x over time. We call a system stable if all points the the domain converge to a steady state (do we need the same steady state?) Notice that is the previous example, the system was stable when $a < 0$, but unstable when $a > 0$.

A simple test for stability is as follows:

$$\text{if } \frac{d\dot{y}}{dy}|_{y^*} < 0, \text{ then the steady state } y^* \text{ is stable}$$

$$\text{if } \frac{d\dot{y}}{dy}|_{y^*} > 0, \text{ then the steady state } y^* \text{ is unstable}$$

Example:

Let $\dot{y} = y^2 - 4y + 3$. Determine the steady states and their stability.

Solution: Let $\dot{y} = 0$. Then $0 = y^2 - 4y + 3 = (y - 1)(y - 3)$. Therefore, we have two steady states, $y = 1$ and $y = 3$.

Next, take the derivative of $\dot{y} = y^2 - 4y + 3$ with respect to y to get

$$\frac{d\dot{y}}{dy} = 2y - 4.$$

Evaluated at $y = 1$, we have

$$\frac{d\dot{y}}{dy}|_{y^*} = 2(1) - 4 = -2 < 0,$$

therefore $y = 1$ is a stable steady state. Evaluated at $y = 3$, we have

$$\frac{d\dot{y}}{dy}|_{y^*} = 2(3) - 4 = 2 > 0,$$

and we have that $y = 3$ is an unstable steady state.

Homework

1. Find the general solutions to the following equations.

   (a) $\dot{y} - 2y = 1$

   (b) $2\dot{y} + 5y = 2$

   (c) $\dot{y} - 2y = 1 - 2x$

   (d) $x\dot{y} - 4y = -2nx$

   (e) $\dot{y} = e^x y$

2. Find the general solutions to the previous questions given that $y_0 = 1$.

3. Draw the phase diagram for the equation $\dot{k}(t) = f(k) - c - (\delta + n)k$, where $f(k) > 0$ when $k > 0$, $f(k) = 0$ when $k = 0$, and $f(k)$ is a concave function which intersects the line $c + (d + n)k = f(k)$ at two points. Also draw the phase diagram for $\frac{\dot{c}(t)}{c} = \frac{1}{\sigma}(f'(k) - \theta - n)$, where $\sigma, \theta, n \in \mathbb{R}$, and $f(k)$ is the same as before. Check the stability of each of the equations. Find their general solutions. Graph k and c is separate graphs with respect to time on the horizontal axis.

Old Markov Chain Homework

1. (midterm exam) You have the following transition probability matrix of a discrete state Markov chain:

$$\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

   What is the probability that the system is in state 2 at time $n + 2$ given that it was in state 3 at time n?

2. (Homework problem) Suppose that the weather can be sunny or cloudy and the weather conditions on succesive mornings form a Markov chain with stationary transition probabilities. Suppose that the transition matrix is as follows

$$\begin{pmatrix} .7 & .3 \\ .6 & .4 \end{pmatrix}$$

   where sunny is state 1 and cloudy is state 2. If it is cloudy on a given day, what is the probability that it will also be cloudy the next day?

3. (Homework problem) Suppose that three boys 1, 2, and 3 are throwing the ball to one another. Whenever 1 has the ball, he throws it to 2 with a probability of 0.2. Whenever 2 has the ball, he will throw it to 1 with probability 0.6. Whenever 3 has the ball, he is equally likely to throw it to 1 or 2.

   (a) Construct the transition probability matrix

   (b) If each of the boys is equally likely to have the ball at a certain time n, which boy is most likely to have the ball at time $n + 2$?