Math Camp Notes: Integration I

Definite Integrals

Consider a function $f(x)$. The area under the graph of the function between points $x = a$ and $x = b$ is denoted by $\int_a^b f(x)dx$, and is called the definite integral of $f(x)$ between a and b. If $f(t)$ and $g(t)$ are integrable functions, then the following properties of the definite integral hold:

1. $\int_a^b [f(t) + g(t)]\, dt = \int_a^b f(t)dt + \int_a^b g(t)dt$
2. $\int_a^b \lambda f(t)dt = \lambda \int_a^b f(t)dt$
3. $\int_a^c f(t)dt = \int_a^b f(t)dt + \int_b^c f(t)dt$
4. $\int_a^b f(t)dt = -\int_b^a f(t)dt$
5. $\int_a^a f(t)dt = 0$

Indefinite Integrals

If $f(x)$ is given then any function $F(x)$ such that $F'(x) = f(x)$ is called an indefinite integral of $f(x)$, or the anti-derivative. Note that there are infinitely many anti-derivatives of a function $f(x)$ since they can differ by a constant. We denote the anti-derivative by $\int f(x)dx$. The following are some simple rules for finding anti-derivatives:

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$
2. $\int e^x dx = e^x + C$
3. $\int a^x \ln(a)dx = a^x + C$
4. $\int f'(x)e^{f(x)}dx = e^{f(x)} + C$
5. $\int \frac{f'(x)}{f(x)}dx = \ln[f(x)] + C$

The Fundamental Theorem of Calculus

If $f(x)$ is continuous on $[a, b]$, and $F(x)$ is the anti-derivative of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(x)$.

Example:

Find the area under the curve $f(x) = x^2$ in the region $[1, 2]$.

By rule one given above, we know that the anti-derivative $F(x) = \frac{1}{3}x^3$. Therefore, the area under the curve is $F(b) - F(a) = \frac{1}{3} \cdot 2^3 - \frac{1}{3} \cdot 1^3 = \frac{7}{3}$.

Example:

Find the derivative of $f(x) = \frac{x^2}{(x+3)(x+2)}$.

We could use the quotient rule as described yesterday morning, but we would have to substitute $u = x^2$ and $v = (x + 3)(x + 2)$. This is likely to give us a big mess. But we can use rule five to calculate it easier.

By rule five,

$$\int \frac{f'(x)}{f(x)}dx = \ln[f(x)].$$

By the definition of an anti-derivative, we know that

$$\frac{d}{dx}\int \frac{f'(x)}{f(x)}dx = \frac{d}{dx}\ln[f(x)] = \frac{f'(x)}{f(x)},$$

which implies

$$f'(x) = f(x) \cdot \frac{d}{dx} \ln[f(x)].$$

It is easier to derivate the log of the function and multiply it by $f(x)$ than it is to use the quotient rule in this case.

$$f'(x) = \frac{x^2}{(x+3)(x+2)} \cdot \frac{d}{dx} \left(2\ln(x) - \ln(x+3) - \ln(x+2)\right) = \frac{x^2}{(x+3)(x+2)} \cdot \left(\frac{2}{x} - \frac{1}{x+3} - \frac{1}{x+2}\right)$$

$$= \frac{2x}{(x+3)(x+2)} - \frac{x^2}{(x+3)^2(x+2)} - \frac{x^2}{(x+3)(x+2)^2}$$

Chain Rule in Reverse or U Substitution

Indefinite Integrals

Suppose $y = g(f(x^*))$. The chain rule states that $\frac{dy}{dx} = \frac{dg}{df}(f(x^*) \cdot \frac{df}{dx}(x^*)$. By the definition of the anti-derivative, we have

$$\int \frac{dy}{dx} dx = y = \int \frac{dg}{df}(f(x^*)) \cdot \frac{df}{dx}(x^*) dx = g(f(x^*)).$$

If we define $f(x) = u$, then we have a technique commonly referred to as u substitution.

Example:

Find $\int (x+1)^{10} dx$.

Let $x + 1 = u$. This implies that $dx = du$ by implicit differentiation. Substituting into the above equation, we now have the easy to solve integral $\int u^{10} du = \frac{u^{11}}{11}$. Substituting back in for u we find that $\int (x+1)^{10} dx = \frac{(x+1)^{11}}{11}$. This is much easier than expanding the function $(x+1)^{10}$ and finding its integral.

Definite Integrals

When using u substitution with definite integrals, one must be careful that the correct limits of integration are used. If we apply a function $u = f(x)$, then we must apply the same function $f(x)$ to the limits of integration. Therefore, if the integral initially reads

$$\int_a^b h(x) dx$$

and we apply a function $u = f(x)$ for easier integration, the integral changes to

$$\int_{f(a)}^{f(b)} g(u) du.$$

Example:

Find $\int_e^{e^2} \frac{1}{x} \cdot \left[\frac{1}{\ln(x)}\right]^3 dx$

Let $u = \ln(x)$. Therefore $du = \frac{1}{x} dx$, the upper limit of integration is $\ln(e^2) = 2$, and the lower limit of integration is $\ln(e) = 1$. The new integral reads

$$\int_1^2 u^{-3} du = -\frac{1}{2} u^{-2}|_1^2 = -\frac{1}{2}[2^{-2} - 1^{-2}] = \frac{3}{8}.$$

Integration by Parts

Given two functions $u(x)$ and $v(x)$, the product rule states that

$$\frac{d}{dx}\left[u(x)\cdot v(x)\right] = u'(x)\cdot v(x) + v'(x)\cdot u(x).$$

By the definition of anti-derivative, we can integrate both sides to get

$$u(x)\cdot v(x) = \int u(x)\cdot v'(x)dx + \int v(x)\cdot u'(x)dx.$$

Subtracting $\int v(x)\cdot u'(x)dx$ from both sides and rearranging, we get integration by parts:

$$\int u(x)\cdot v'(x)dx = u(x)\cdot v(x) - \int v(x)\cdot u'(x)dx$$

or in simpler notation:

$$\int u\cdot dv = u\cdot v - \int v\cdot du$$

Example:

Find $\int \ln(x)dx$.

Let $u = \ln(x) \Rightarrow du = \frac{1}{x}dx$. Let $dv = dx \Rightarrow v = x$. Then

$$\int u\cdot dv = u\cdot v - \int v\cdot du = x\cdot\ln(x) - \int x\cdot\frac{1}{x}dx = x[\ln(x) - 1].$$

Example:

Find $\int xe^{-x}dx$.

Let $u = x \Rightarrow du = dx$. Let $dv = e^{-x}dx \Rightarrow v = -e^{-x}$. Then

$$\int u\cdot dv = u\cdot v - \int v\cdot du = -xe^{-x} - \int -e^{-x}dx = -xe^{-x} - e^{-x} = -e^{-x}(1+x).$$

Homework

Find the following:

1. $\int 8x^{-5}dx$, $x \neq 0$.
2. $\int (7e^x + 3)dx$
3. $\int \frac{6x}{x^2+13}dx$
4. $\int (x+3)(x+1)^{\frac{1}{2}}$
5. $\int xe^x dx$
6. $\int x^3\sqrt{1+x^2}dx$
7. $\int x^2\ln(x)dx$
8. $\int x^3e^{4x}dx$

Evaluate the following:

1. $\int_0^1 x(x^2+6)dx$
2. $\int_{-1}^1 (ax^2+bx+c)dx$
3. $\int_1^2 e^{-2x}dx$