Math Camp Notes: Integration II

Double and Triple Integration

In order to evaluate the double integral

$$\int_a^b \int_c^d f(x,y)dxdy$$

it is helpful to rewrite the integral as

$$\int_a^b \left(\int_c^d f(x,y)dx \right) dy$$

and evaluate the inside integral first. Then the outside integral can be evaluated to obtain the solution.

Example:

Evaulate $\int_1^2 \int_2^4 (x+y)dxdy$.

$$\int_1^2 \int_2^4 (x+y)dxdy = \int_1^2 \left(\int_2^4 (x+y)dx \right) dy = \int_1^2 \left((\frac{1}{2}x^2 + xy)|_2^4 \right) dy = \int_1^2 (6+2y)\, dy = 6y + y^2|_1^2 = 9$$

Triple integrals (and higher order integrals) are evaluated in the same manner as doubles; we first integrate with respect to one variable and work our way back to lower order integrals.

$$\int_a^b \int_c^d \int_e^f f(x,y,z)dxdydz = \int_a^b \left(\int_c^d \left(\int_e^f f(x,y)dx \right) dy \right) dz$$

Order of Integration

The limits of integration will only be scalar if we are integrating over a rectangle (or its equivalent in higher dimensions). In this case, the simplicity of the order of integration does not matter.

Example:

Recall the double integral $\int_1^2 \int_2^4 (x+y)dxdy$ above which we evaluated with respect to x first. Now evaluate with respect to y first.

$$\int_2^4 \int_1^2 (x+y)dydx = \int_2^4 \left(\int_1^2 (x+y)dy \right) dx = \int_2^4 \left((\frac{1}{2}y^2 + xy)|_1^2 \right) dy =$$

$$= \int_2^4 \left(\frac{3}{2} + x \right) dy = \frac{3}{2}y + \frac{1}{2}y^2|_2^4 = 3 + 6 = 9$$

However, we are integrating with respect to variables that are also in the limits of integration, then different orders of integration influence the difficulty of the problem. To see this, consider the integral of $f(x,y)$ over the region $R = \{1 \le y \le 2,\ y \le x \le y^2\}$. By drawing the area of integration, we can easily see that integrating with respect to x first is much easier. Notice that the region R is equivalent to the region $S = \{1 \le x \le 2,\ x \le y \le \sqrt{x}\} \cup \{2 \le x \le 4,\ 2 \le y \le \sqrt{x}\}$. If I integrate with respect to y first, I must split the integral into two parts, since the bounds of integration in the x direction change at $x = 2$.

Example:

Evaluate $\int_1^2 \int_x^{2x} xy^2 dydx$ with respect to y first:

$$\int_1^2 \int_x^{2x} xy^2 dydx = \int_1^2 (\frac{1}{3}xy^3|_x^{2x})dx = \int_1^2 \frac{7}{3}x^4 dx = \frac{7}{15}x^5|_1^2 = \frac{7}{15}(2^5 - 1^5) = \frac{217}{15}$$

Now evaluate with respect to x first:

$$\int_1^2 \int_x^{2x} xy^2 dydx = \int_1^2 \int_1^y xy^2 dxdy + \int_2^4 \int_{\frac{y}{2}}^2 xy^2 dxdy = \int_1^2 \frac{1}{2}x^2y^2|_1^y dy + \int_2^4 \frac{1}{2}x^2y^2|_{\frac{y}{2}}^2 dy =$$

$$= \int_1^2 (\frac{1}{2}y^4 - \frac{1}{2}y^2)dy + \int_2^4 (2y^2 - \frac{y^4}{8})dy = (\frac{1}{10}y^5 - \frac{1}{6}y^3)|_1^2 + (\frac{2}{3}y^3 - \frac{1}{40}y^5)|_2^4 =$$

$$= \frac{31}{10} - \frac{7}{6} + \frac{112}{3} - \frac{992}{40} = \frac{217}{15}$$

Transformations

Consider the integral $\int_a^b \int_c^d f(x,y)$. As we have seen previously, integrals are sometimes easier to evaluate when we substitute a variable u in for a function of x and y. Consider the following integral:

$$\int_2^3 \int_0^{x-2} \frac{1}{(x+y)(x-y)}$$

We can guess that making a substitution $u = x + y$ and $v = x - y$ would make the integration a lot simpler. However, there is more than meets the eye in making this transformation. Not only will the limits of integration change, but we must also change the function itself beyond just making the substitution.

In general, say we are given $f(x,y)$, a region of integration, and two transformation functions $u(x,y)$ and $v(x,y)$. In order to integrate using the transformation, we first solve the system of transformation functions for $x(u,v)$ and $y(u,v)$. Then me must compute the Jacobian matrix

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

and find the absolute value of its determinant. We will cover determinants later, but the absolute value of the determinant of this specific matrix is $\left|\frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}\right| = |J|$. We then go back to the original equation $f(x,y)$, plug in $x(u,v)$ and $y(u,v)$ to get a new equation $g(u,v)$ and multiply it by the Jacobian to get our new integrand

$$\int\int g(u,v)\,|J|\,dudv.$$

Finally, we must change the limits of integration. This is done simply by looking at the possible values x and y took on, and then seeing what that would correspond to in u and v. For example, if $x, y \in (0,1)$ and $u = xy$ and $v = \frac{x}{y}$, we can see that $u \in (0,1)$ and $v \in (0,\infty)$. We then integrate over that region.

Example:

Transform $\int_2^3 \int_0^{x-2} \frac{1}{(x+y)(x-y)} dydx$ using $u = x + y$ and $v = x - y$.

Solving the system for x and y we get $x = \frac{u+v}{2},\ y = \frac{u-v}{2}$. The jacobian of this system is

$$\left|\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\right| = \frac{1}{2}$$

Plugging this into the original equation, we have

$$\int\int \frac{1}{(\frac{u+v}{2} + \frac{u-v}{2})(\frac{u+v}{2} - \frac{u-v}{2})} \cdot \frac{1}{2}dudv = \int\int \frac{1}{2uv}dudv$$

To find the new limits of integration, we notice that there are three boundaries to the original integral: $x \leq 3$, $y \geq 0$, $y \leq x - 2$. Substituting in for and x and y in each of these three, we get:

$$\frac{u+v}{2} \leq 3 \Rightarrow u + v \leq 6$$

$$\frac{u-v}{2} \geq 0 \Rightarrow u \geq v$$

$$\frac{u-v}{2} \leq \frac{u+v}{2} - 2 \Rightarrow v \geq 2$$

Plotting these three, we see that the area of integration in $u,\ v$ space is now an isolceles triangle with corners at $(2,2)$, $(3,3)$, and $(4,2)$. We can then set up the orders of integration so the integral now reads

$$\int_2^3 \int_v^{6-v} \frac{1}{2uv} du dv.$$

This is hard to integrate, so we will leave it for the interested student to do at home.

Homework

Evaluate the following integrals:

1. $\int_0^2 \int_0^{4-x^2} xy dy dx$
2. $\int_0^1 \int_{1-x}^{\sqrt{1-x^2}} x^2 y dy dx$
3. $\int \int x dy dx$ for the region bounded by $y = 2x$ and $y = 3 - x^2$
4. $\int_0^1 \int_y^1 x^2 \sin(xy) dx dy$

Sketch the following regions:

1. $1 \leq x \leq 2,\ 2 \leq y \leq 7$
2. $0 \leq x \leq 2,\ \frac{x^2}{2} \leq y \leq x$
3. $1 \leq x \leq 2,\ x^2 \leq y \leq x + 2$
4. $0 \leq x \leq 1,\ x^2 \leq y \leq x$

Integrate the following:

1. $f(x,y) = \sin(x^2)$ over the area $0 \leq x \leq 2,\ 0 \leq y \leq \frac{x}{2}$ with respect to x first
2. $f(x,y) = \sin(x^2)$ over the area $0 \leq x \leq 2,\ 0 \leq y \leq \frac{x}{2}$ with respect to y first

In 1st semester econometrics, you will be asked to integrate probability density functions (or p.d.f.s) in order to find the probability that certain events will occur. Sometimes it is necessary to transform the random variables. For example, if we are given the density functions for X_1 and X_2, and Y_1 and Y_2 as functions of X_1 and X_2, we can then transform this system to solve for the p.d.f.'s of Y_1 and Y_2. The following problems are taken from p.d.f.'s in the first semester econometrics textbook, section 3.7. For each of the following,

- Find X_1 and X_2 as functions of Y_1 and Y_2
- Find the determinant of the Jacobian
- Sketch the region of integration in terms of X_1 and X_2
- Sketch the region of integration in terms of Y_1 and Y_2
- Evaluate the new integral in terms of Y_1 and Y_2

1. $f(X_1, X_2) = e^{-X_1 - X_2}$ over the region $X_1 > 0,\ X_2 > 0$, where $Y_1 = \frac{X_1}{X_1 + X_2}$ and $Y_2 = X_1 + X_2$.
2. $f(X_1, X_2) = 8X_1X_2$ over the region $0 \leq X_1 \leq X_2 \leq 1$, where $Y_1 = \frac{X_1}{X_2}$ and $Y_2 = X_2$.

Hint: As a check, realize that all p.d.f.'s integrate to one. So the original integrals with respect to X_1 and X_2, as well as the transformed integral with respect to Y_1 and Y_2, should integrate to one. Try it!