Math Camp Notes: Kuhn-Tucker Theorem

The most common problems in economics are maximization problems dealing with only inequality constraints. Many of these constraints come in the form of non-negativity constraints, such as requiring consumption to be weakly positive. Consider the following problem:

$$\max_{\mathbf{x}\in\mathbb{R}^n} f(x_1,\ldots,x_n)$$

$$\text{subject to } g_1(x_1,\ldots,x_n),\ldots,g_m(x_1,\ldots,x_n)$$

$$\text{and } x_1 \geq 0,\ldots,x_n \geq 0$$

The lagrangean function we would write would take the form

$$L(\mathbf{x},\lambda,\mathbf{v}) = f(\mathbf{x}) - \sum_{i=1}^{m}(g_i(\mathbf{x})-b_i) + \sum_{i=1}^{n} v_i x_i,$$

which would lead to the following first order conditions:

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \sum_{i=1}^{m}\lambda_i\frac{\partial g_i}{\partial x_1} + v_1 = 0,\ldots,\frac{\partial L}{\partial x_n} = \frac{\partial f}{\partial x_n} - \sum_{i=1}^{m}\lambda_i\frac{\partial g_i}{\partial x_n} + v_n = 0$$

$$\lambda_1\left[g_1(\mathbf{x})-b_1\right] = -\lambda_1\frac{\partial L}{\partial\lambda_1} = 0,\ldots,\lambda_m\left[g_m(\mathbf{x})-b_m\right] = -\lambda_m\frac{\partial L}{\partial\lambda_m}$$

$$v_1x_1 = 0,\ldots,v_nx_n = 0, \text{ and } \lambda_i, x_j \geq 0\ \forall\ i=1,\ldots,m \text{ and } \forall\ j=1,\ldots,n$$

That's a lot of conditions! $(2m+3n)$

Now consider the lagrangean without the nonnegativity constraints, and call it the Kuhn-Tucker lagrangean:

$$L(\mathbf{x},\lambda,\mathbf{v}) = \tilde{L}(\mathbf{x},\lambda) + \sum_{i=1}^{n} v_i x_i$$

The first n first order conditions can be rewritten as

$$\frac{\partial L}{\partial x_j} = \frac{\partial\tilde{L}}{\partial x_j} + v_j = 0\ \forall\ j=1,\ldots,n,$$

which implies

$$\frac{\partial\tilde{L}}{\partial x_j} = -v_j\ \forall\ j=1,\ldots,n.$$

Plugging those into the nonnegativity constraints we have that

$$x_j\frac{\partial\tilde{L}}{\partial x_j} = 0 \text{ and } \frac{\partial\tilde{L}}{\partial x_j} \leq 0.$$

Also notice that

$$\frac{\partial L}{\partial\lambda_j} = \frac{\partial\tilde{L}}{\partial\lambda_j} = b_j - g_j(\mathbf{x}) \geq 0\ \forall\ j=1,\ldots,m,$$

which implies

$$\frac{\partial\tilde{L}}{\partial\lambda_j} \geq 0 \text{ and } \lambda_j\frac{\partial\tilde{L}}{\partial\lambda_j} = 0\ \forall\ j=1,\ldots,m.$$

In summary, the following conditions give us the Kuhn-Tucker lagrangean:

$$\frac{\partial\tilde{L}}{\partial x_j} \leq 0 \text{ and } x_j\frac{\partial\tilde{L}}{\partial x_j} = 0\ \forall\ j=1,\ldots,n$$

$$\frac{\partial\tilde{L}}{\partial\lambda_j} \geq 0 \text{ and } \lambda_j\frac{\partial\tilde{L}}{\partial\lambda_j} = 0\ \forall\ j=1,\ldots,m$$

This is $2n+2m$ constraints, n less than before.