Math Camp Notes: Linear Algebra I

Basic Matrix Operations and Properties

Consider two $n \times m$ matrices:

$$A = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix},\ B = \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nm} \end{pmatrix}$$

Then the basic matrix operations are as follows:

1. $A + B = \begin{pmatrix} a_{11}+b_{11} & \dots & a_{1m}+b_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & \dots & a_{nm}+b_{nm} \end{pmatrix}$
2. $\lambda A = \begin{pmatrix} \lambda a_{11} & \dots & \lambda a_{1m} \\ \vdots & \ddots & \vdots \\ \lambda a_{n1} & \dots & \lambda a_{nm} \end{pmatrix}$, where $\lambda \in \mathbb{R}$

Notice that the elements in the matrix are numbered such that $a_{i,j}$, where i is the row and j is the column in which the element $a_{i,j}$ is found.

In order to multiply matricies CD, the number of columns in the C matrix must be equal to the number of rows in the D matrix. Say C is an $n \times m$ matrix, and D is an $m \times k$ matrix. Then multiplication is defined as follows:

$$E = \underbrace{C}_{n\times m}\underbrace{D}_{m\times k} = \underbrace{\begin{pmatrix} \sum_{q=1}^m c_{1,q}d_{q,1} & \dots & \sum_{q=1}^m c_{1,q}d_{q,k} \\ \vdots & \ddots & \vdots \\ \sum_{q=1}^m c_{n,q}d_{q,1} & \dots & \sum_{q=1}^m c_{n,q}d_{q,k} \end{pmatrix}}_{n\times k}$$

There are two notable special cases for multiplaction of matricies. The first is called the inner product, which occurs when two vectors of the same length are multiplied together such that the result is a scalar:

$$v \cdot z = \underbrace{v}_{1\times n}\underbrace{z'}_{n\times 1} = \begin{pmatrix} v_1 & \dots & v_n \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \sum_{i=1}^n v_i z_i$$

The second is called the outer product:

$$\underbrace{v'}_{n\times 1}\underbrace{z}_{1\times n} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \begin{pmatrix} v_1 & \dots & v_n \end{pmatrix} = \underbrace{\begin{pmatrix} z_1v_1 & \dots & z_1v_n \\ \vdots & \ddots & \vdots \\ z_nv_1 & \dots & z_nv_n \end{pmatrix}}_{n\times n}.$$

Note that when we multiplied the matricies C and D together, the resulting $e_{i,j}$th element of E was just the inner product of the ith row of C and jth column of D. Also, note that even if two matricies X and Y are both $n \times n$, then $XY \neq YX$, except in special cases.

Types of Matricies

Zero Matricies

A zero matrix is a matrix where each element is 0

$$\mathbf{0} = \underbrace{\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}}_{n\times k}$$

The following properties hold for zero matricies:

1. $A + \mathbf{0} = A$
2. If $AB = \mathbf{0}$, it is not necessarily the case that $A = \mathbf{0}$ or $B = \mathbf{0}$.

Identity Matricies

The identity matrix is a matrix with zeroes everywhere except along the diagonal. Note that the number of columns must equal the number of rows.

$$I = \underbrace{\begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}}_{n \times n}$$

The reason it is called the identity matrix is because $AI = IA = A$.

Square, Symmetric, and Transpose Matricies

A square matrix is a matrix whose number of rows is the same as its number of columns. For example, the identity matrix is always square. If a square matrix has the property that $a_{i,j} = a_{j,i}$ for all its elements, then we call it a symmetric matrix.

The transpose of a matrix A, denoted A' is a matrix such that for each element of A', $a'_{i,j} = a_{j,i}$. For example, the transpose of the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

is

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}.$$

Note that a matrix A is symmetric if $A = A'$.

The following properties of the transpose hold:

1. $(A')' = A$.
2. $(A + B)' = A' + B'$.
3. $(\alpha A)' = \alpha A'$.
4. $(AB)' = B'A'$.
5. If the matrix A is $n \times k$, then A' is $k \times n$.

Diagonal and Triangular Matricies

A square matrix A is diagonal if it is all zeroes except along the diagonal:

$$\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

Note that all diagonal matricies are also symmetric.

A square matrix A is upper triangular if all of its entries below the diagonal are zero

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix},$$

and is lower triangular if all its entries above the diagonal are zero

$$\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

Inverse Matricies

If there exists a matrix B such that $BA = AB = I$, then we call B the inverse of A, and denote it A^{-1}. Note that A only has an inverse if it is a square matrix. The following properties of inverses hold:

1. $\left(A^{-1}\right)^{-1} = A$
2. $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$
3. $(AB)^{-1} = B^{-1}A^{-1}$ if B^{-1}, A^{-1} exist.
4. $(A')^{-1} = \left(A^{-1}\right)'$

Other Important Matricies

A matrix A is orthogonal if $A'A = I$. In other words, a matrix is orthogonal if it is its own inverse. A matrix is idempotent if it is both symmetric and $AA = A$. Orthogonal and idempotent matricies are especially used in econometrics.

Determinants

We define the determinant inductively.

1. The determinant of a 1×1 matrix $[a]$ is a, and is denoted $\det(a)$.
2. The determinant of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad-bc$. Notice that this is the same as $a\cdot\det(d)-b\cdot\det(c)$. The first term is the (1,1)th entry of A times the determinant of that submatrix obtained by deleting from A the row and column which contain that entry. The second term is (1,2)th entry times the determinant of the submatrix obtained by deleting A from the row and column which contain that entry. The terms alternate in sign, with teh first term being added and the second term being subtracted.
3. The determinant of a 3×3 matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is $aei + bfg + cdh - ceg - bdi - afh$. Notice that this can be written as $a \cdot \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \cdot \det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \cdot \det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$.

Can you see the pattern? In order to obtain the determinant, we multiply each element in the top row with the determinant of the matrix left when we delete the row and column in which the respective elements reside. The signs of the terms alternate, starting with postive.

In order to write the definition of the determinant of an nth order matrix, it is useful to define the (i,j)th **minor** of A and the (i,j)th **cofactor** of A:

- Let A be an $n \times n$ matrix. Let $A_{i,j}$ be the $(n-1) \times (n-1)$ submatrix obtained by deleting row i and column j from A. Then the scalar $M_{ij} = \det(A_{ij})$ is called the (i,j)th **minor** of A.
- The scalar $C_{ij} = (-1)^{i+j} M_{ij}$ is called the (i,j)th **cofactor** of A. The cofactor is merely the signed minor.

Armed with these two definitions, we notice that the determinant for the 2×2 matrix is

$$\det(A) = aM_{11} - bM_{12} = aC_{11} + bC_{12},$$

and the determinant for the 3×3 matrix is

$$\det(A) = aM_{11} - bM_{12} - cM_{13} = aC_{11} + bC_{12} + C_{13}.$$

Therefore, we can define the determinant for an $n \times n$ sqaure matrix as follows:

$$\det\left(\underbrace{A}_{n \times n}\right) = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}.$$

Notice that the definition of the determinant uses elements and cofactors for the top row only. This is called a **cofactor expansion along the first row**. However, a cofactor expansion along any row or column will be equal to the determinant. The proof of this asssertion is left as a homework problem for the 3×3 case.

Example: Find the determinant of the upper diagonal matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}$$

Answer: The determinant is:

$$aC_{11} + bC_{12} + C_{13} = a \cdot \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \cdot \det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \cdot \det\begin{pmatrix} d & e \\ g & h \end{pmatrix} =$$

$$= 1 \cdot \det\begin{pmatrix} 3 & 0 \\ 5 & 6 \end{pmatrix} - 0 \cdot \det\begin{pmatrix} 2 & 0 \\ 4 & 6 \end{pmatrix} + 0 \cdot \det\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = 1 \cdot 3 \cdot 6 = 18$$

Now lets expand along the second column instead of the third row:

$$aC_{12} + bC_{22} + C_{32} = -b \cdot \det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + e \cdot \det\begin{pmatrix} a & c \\ g & i \end{pmatrix} - h \cdot \det\begin{pmatrix} a & c \\ d & f \end{pmatrix} =$$

$$= -0 \cdot \det\begin{pmatrix} 2 & 0 \\ 4 & 6 \end{pmatrix} + 3 \cdot \det\begin{pmatrix} 1 & 0 \\ 4 & 6 \end{pmatrix} - 5 \cdot \det\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = 3 \cdot 6 = 18$$

Singular and Non-singular Matrices

Consider the system of equations:

$$y_1 = a_1x_1 + b_1x_2 + \ldots + c_1x_n$$

$$y_2 = a_2x_1 + b_2x_2 + \ldots + c_2x_n$$

$$\vdots$$

$$y_m = a_2x_1 + b_2x_2 + \ldots + c_2x_n$$

The functions can be written in matrix form by

$$\underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}}_{m\times 1} = \underbrace{\begin{pmatrix} a_1 & b_1 & \dots & c_1 \\ a_2 & b_2 & \dots & c_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_m & b_m & \dots & c_m \end{pmatrix}}_{m\times n} \cdot \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{n\times 1}$$

In short, we can write this system as $\mathbf{y} = A\mathbf{x}$. For simplicity, denote the ith row in the matrix as a separate vector v_i, so that

$$A = \underbrace{\begin{pmatrix} a_1 & b_1 & \dots & c_1 \\ a_2 & b_2 & \dots & c_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_m & b_m & \dots & c_m \end{pmatrix}}_{m\times n} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}.$$

We say the vectors $v_1, \dots, v_m$ are **linearly dependant** if there exist scalars $q_1, \dots, q_m$, not all zero, such that:

$$\sum_{i=1}^{m} q_i v_i = 0.$$

We say the vectors $v_1, \dots, v_m$ are **linearly independant** if the only scalars $q_1, \dots, q_m$ such that:

$$\sum_{i=1}^{m} q_i v_i = 0$$

are $q_1 = \dots = q_m = 0$. We can use this definiton of linear independance and dependance for columns as well.

The **rank** of a matrix is the number of linearly independant rows or columns in a matrix (Note: the number of linearly independant rows are the same as the number of linearly independant columns).

If we draw the lines in the system in $\mathbb{R}^N$, then they will all cross either at no points, at one point, or at infinitely many points. Therefore, the system may have no solutions, one solution, or many solutions for a given vector $\mathbf{y}$. If it has more than one solution, it will have infinitely many solutions, since straight lines can only intersect once if they do not coincide. The following are conditions on the number of solutions of a system:

1. A system of linear equations with coefficient matrix A will have a solution for each choice of $\mathbf{y}$ iff rank $A =$ the number of rows in A.

2. A system of linear equations with coefficient matrix A will have at most one solution for each choice of $\mathbf{y}$ iff rank $A =$ the number of columns in A.

This implies that a system of linear equations with coefficient matrix A will have exactly one solution for each choice of $\mathbf{y}$ iff rank $A =$the number of columns in $A =$ the number of rows of A. If a square matrix has exactly one solution for each $\mathbf{y}$, we say the matrix is **non-singular**. Otherwise, it is non-singular and has infinitely many solutions.

Elementary Row Operations

Recall the system of equations:

$$y_1 = a_1x_1 + b_1x_2 + \dots + c_1x_n$$

$$y_2 = a_2x_1 + b_2x_2 + \dots + c_2x_n$$

$$\vdots$$

$$y_m = a_2x_1 + b_2x_2 + \ldots + c_2x_n$$

which can be written in matrix form by

$$\underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}}_{m\times 1} = \underbrace{\begin{pmatrix} a_1 & b_1 & \ldots & c_1 \\ a_2 & b_2 & \ldots & c_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_m & b_m & \ldots & c_m \end{pmatrix}}_{m\times n} \cdot \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{n\times 1},$$

or $\mathbf{y} = A\mathbf{x}$. There are three types of elementary row operations we can perform on this matrix without changing the solution set:

1. Interchanging two rows of the matrix
2. Multiplying each element of a row by the same non-zero scalar.
3. Change a row by adding it to a multiple of another row.

You should convince yourself that these three operations do not change the solution set of the matrix.

Using Elementary Row Operations to Check Linear Independence

A row of a matrix is said to have k **leading zeros** if the $(k+1)$th element of the row is non zero while the first k elements are zero. A matrix is in **row echelon form** if each row has strictly more leading zeros that the preceding row. For example, the matrices

$$A = \begin{pmatrix} 4 & 2 \\ 0 & 7 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 & 10 \\ 0 & 8 & -3 \\ 0 & 0 & 6 \end{pmatrix}$$

are in row echelon form while the matrices

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 6 & 7 \\ 0 & 0 & 8 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 1 \\ 2 & 4 \end{pmatrix}$$

are not. However, B would be in row echelon form if we performed the elementary matrix operation of switching the two rows.

In order to get the matrix into row echelon form, we can perform elementary matrix operations on the rows. For example, consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 8 & 6 \\ 1 & 1 & 1 \end{pmatrix}.$$

By taking the third row and subtracting it from the first row, we obtain the matrix

$$A_1 = \begin{pmatrix} 0 & 1 & 2 \\ 4 & 8 & 6 \\ 1 & 1 & 1 \end{pmatrix}$$

We can also subtract four times the third row from the second row

$$A_2 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 4 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

Now subtract four times the first row from the second row to obtain

$$A_3 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & -6 \\ 1 & 1 & 1 \end{pmatrix}$$

Then rearrange to get the matrix in row echelon form

$$A_4 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -6 \end{pmatrix}$$

When a matrix is in row echelon form, it is easy to check whether all the rows are linearly independant. For linear independance, we must have that all the rows in the row echelon form are non-zero. If not, then the matrix will have linearly dependant rows. We have shown that all the rows in A are linearly independant because the row echelon form contains no zero rows.

Another way of defining the rank of matrix is by the number of non-zero (or linearly independant) rows in a matrix. For example, the matrix A is of full rank since all its rows are linearly independant.

Using Elementary Row Operations to Solve a System of Equations

We can solve a system of equations by writing the matrix

$$\begin{pmatrix} a_1 & b_1 & \dots & c_1 & | & y_1 \\ a_2 & b_2 & \dots & c_2 & | & y_2 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots \\ a_m & b_m & \dots & c_m & | & y_m \end{pmatrix},$$

called the **augmented matrix** of A, and use elementary row operations.

Example:

Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 8 & 6 \\ 1 & 1 & 1 \end{pmatrix}$$

as before. Say that the vector $\mathbf{y} = (1, 1, 1)'$. Then the augmented matrix is

$$\begin{pmatrix} 1 & 2 & 3 & | & 1 \\ 4 & 8 & 6 & | & 1 \\ 1 & 1 & 1 & | & 1 \end{pmatrix}$$

Performing the same matrix operations as before, we have

$$\begin{pmatrix} 1 & 2 & 3 & | & 1 \\ 4 & 8 & 6 & | & 1 \\ 1 & 1 & 1 & | & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 2 & | & 0 \\ 4 & 8 & 6 & | & 1 \\ 1 & 1 & 1 & | & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 2 & | & 0 \\ 0 & 4 & 2 & | & -3 \\ 1 & 1 & 1 & | & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 2 & | & 0 \\ 0 & 0 & -6 & | & -3 \\ 1 & 1 & 1 & | & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & -6 & | & -3 \end{pmatrix}$$

We continue the row operations until the left hand side of the augmented matrix looks like the identity matrix:

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 1 & | & \frac{1}{2} \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & \frac{1}{2} \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$$

Notice that this implies

$$\begin{pmatrix} \frac{3}{2} \\ -1 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \mathbf{x} = \left(\frac{3}{2}, -1, \frac{1}{2}\right),$$

so we have found a solution using elementary row operations.

In summary, if we form the augmented matrix of A, reduce the left hand side of the matrix to is **reduced row echelon form** (so that each row contains all zeros, except for the possibility of a one in a column of all zeros) through elementary row operations, then the remaining vector on the right hand side will be the solution to the system.

Inverse Matricies

It can be shown that the inverse of a square matrix will exist if it's determinant is non-zero. The following statements are all equivalent for a square matrix A:

1. A is non-singular.
2. All the columns and rows in A are linearly independent.
3. A has full rank.
4. Exactly one solution X^* exists for each vector Y^*.
5. A is invertible.
6. $\det(A) \neq 0$.
7. The row-echelon form of the matrix is upper triangular.
8. The reduced row echelon form is the identity matrix.

Calculating the Inverse Matrix by the Adjoint Matrix

The **adjoint matrix** of a square matrix A is the transposed matrix of cofactors of A, or

$$\text{adj}(A) = \begin{pmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{pmatrix}$$

Notice that the adjoint of a 2×2 matrix is $\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$.

The inverse of the matrix A can be found by

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A).$$

Therefore, the inverse of a 2×2 matrix is

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \cdot \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

Calculating the Inverse Matrix by the Elementary Row Operations

To calculate the inverse matrix, form the augmented matrix

$$\left(\begin{array}{cccc|cccc} a_{11} & a_{12} & \dots & a_{1n} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & 0 & 0 & \dots & 1 \end{array}\right),$$

where the left hand side is the matrix A and the right hand side is the identity matrix. Reduce the left hand side to the reduced row echelon form, and what remains on the right hand side will be the inverse matrix of A. In other words, by elementary row operations, you can transform the matrix $(A|I)$ to the matrix $(I|A^{-1})$.

Kramer's Rule

Let A be a non-singular matrix. Then the unique solution $\mathbf{x} = (x_1, \dots, x_n)$ of the $n \times n$ system $\mathbf{y} = A\mathbf{x}$ is:

$$x_i = \frac{\det(B_i)}{\det(A)} \text{ for } i = 1, \dots, n,$$

where B_i is the matrix A with the right hand side $\mathbf{y}$ replacing the ith column of A.

Example:

Consider the linear IS-LM model

$$sY + ar = I^0 + G$$

$$mY - hr = M_s - M^0$$

where Y is the net national product, r is the interest rate, s is the marginal propensity to save, a is the marginal efficiency of capital, $I = I^0 - ar$ is investment, m is money balances needed per dollar of transactions, G is government spending, and M_s is the money supply. All the parameters are postive. We can rewrite the system as

$$\begin{pmatrix} I^0 + G \\ M_s - M^0 \end{pmatrix} = \begin{pmatrix} s & a \\ m & -h \end{pmatrix} \cdot \begin{pmatrix} Y \\ r \end{pmatrix}.$$

By Kramer's rule, we have

$$Y = \frac{\begin{vmatrix} I^0 + G & a \\ M_s - M^0 & -h \end{vmatrix}}{\begin{vmatrix} s & a \\ m & -h \end{vmatrix}} = \frac{(I^0 + G)h + a(M_s - M^0)}{sh + am}$$

$$r = \frac{\begin{vmatrix} s & I^0 + G \\ m & M_s - M^0 \end{vmatrix}}{\begin{vmatrix} s & a \\ m & -h \end{vmatrix}} = \frac{(I^0 + G)m - s(M_s - M^0)}{sh + am}$$

Principal Minors and Leading Principle Minors

Principal Minors

Let A be an $n \times n$ matrix. A $k \times k$ submatrix of A obtained by deleting any $n - k$ columns and the same $n - k$ rows from A is called a kth-order **principle submatrix** of A. The determinant of a $k \times k$ principal submatrix is called a kth order **principle minor** of A.

Example:

List all the principle minors of the 3×3 matrix:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Answer: There is one third order principle minor of A, $\det(A)$. There are three second order principal minors:

1. $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, formed by deleting the third row and column of A.

2. $\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$, formed by deleting the second row and column of A.

3. $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, formed by deleting the first row and column of A.

There are also three first order principle minors: a_{11}, by deleting the last two rows and columns; a_{22}, by deleting the first and last rows and columns; and a_{33}, by deleting the first two rows and columns.

Leading Principal Minors

A **leading principal minor** the the determinant of the **leading principal submatrix** obtained by deleting the last $n - k$ rows and columns of an $n \times n$ matrix A.

Example:

List all the leading principle minors of the 3×3 matrix:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Answer: There are three leading principal minors:

1. $|a_{11}|$, formed by deleting the last two rows and columns of A.

2. $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, formed by deleting the last row and column of A.

3. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, formed by deleting the no rows or columns of A.

Why in the world do we care about principal and leading principal minors? We need to calculate the signs of the leading principal minors in order to determine the definiteness of a matrix. We need definiteness to check second-order conditions for maxima and minima. We also need definiteness of the Hessian matrix to check to see whether or not we have a concave function.

Quadratic Forms and Definiteness

Quadratic Forms

Consider the function $F : \mathbb{R}^2 \to \mathbb{R}$, where $F = a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2$. We call this a **quadratic form** in $\mathbb{R}^2$. Notice that this can be expressed in matrix form as

$$F(\mathbf{x}) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} \\ \frac{1}{2}a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{x}'A\mathbf{x},$$

where $\mathbf{x} = (x_1, x_2)$, and A is unique and symmetric.

The quadratic form in $\mathbb{R}^n$ is

$$F(\mathbf{x}) = \sum_{i,j=1}^{n} a_{ij} x_i x_j,$$

where $\mathbf{x} = (x_1, \ldots, x_n)$, and A is unique and symmetric. This can also be expressed in matrix form:

$$F(\mathbf{x}) = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & \frac{1}{2}a_{21} & \ldots & \frac{1}{2}a_{1n} \\ \frac{1}{2}a_{12} & a_{22} & \ldots & \frac{1}{2}a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}a_{n1} & \frac{1}{2}a_{n2} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{x}'A\mathbf{x}.$$

Definiteness

Let A be an $n \times n$ symmetric matrix. Then A is:

1. **positive definite** if $\mathbf{x}'A\mathbf{x} > 0 \ \forall \ \mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$.
2. **positive semidefinite** if $\mathbf{x}'A\mathbf{x} \geq 0 \ \forall \ \mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$.
3. **negative definite** if $\mathbf{x}'A\mathbf{x} < 0 \ \forall \ \mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$.
4. **negative semidefinite** if $\mathbf{x}'A\mathbf{x} \leq 0 \ \forall \ \mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$.
5. **indefinite** if $\mathbf{x}'A\mathbf{x} > 0$ for some $\mathbf{x} \in \mathbb{R}^n$, and < 0 for some other $\mathbf{x} \in \mathbb{R}^n$.

We can test for the definiteness of the matrix in the following fashion:

1. A is positive definite iff all of its n leading principal minors are strictly positive.
2. A is negative definite iff all of its n leading principal minors alternate in sign, where $|A_1| < 0$, $|A_2| > 0$, $|A_3| < 0$, etc.
3. If some kth order leading pricipal minor of A is nonzero but does not fit either of the above sign patterns, then A is indefinite.

If the matrix A would meet the criterion for positive or negative definiteness if we relaxed the strict inequalites to weak inequalities (i.e. we allow zero to fit into the pattern), then although the matrix is not positive or negative definite, it may be positive or negative semidefinite. In this case, we employ the following tests:

1. A is positive semidefinite iff *every* principal minor of A is ≥ 0.
2. A is negative semidefinite iff *every* principal minor of A of odd order is ≤ 0 and every principal minor of even order is ≥ 0.

Notice that for determining semidefiniteness, we can no longer check just the leading principal minors, but we must check *all* principal minors. What a pain!

Homework

Do the following:

1. Let $A = \begin{pmatrix} 2 & 0 \\ 3 & 8 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 2 \\ 6 & 3 \end{pmatrix}$. Find $A - B$, $A + B$, AB, and BA.
2. Let $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $u = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$. Find $u \cdot v$, $u'v$ and $v'u$.

3. Prove that the multiplication of any matrix with its transpose yields a symmetric matrix
4. Prove that A only has an inverse if it is a square matrix.
5. Prove the first four properties of transpose matricies above.
6. In econometrics, we deal with a matrix called the projections matrix: $A = I - X\left(X'X\right)^{-1}X'$. Must A be square? Must $X'X$ be square? Must X be square?
7. Show that the projection matrix in 5 is idempotent.
8. Evaluate the following determinants:

   (a) $\begin{vmatrix} 1 & 1 & 4 \\ 8 & 11 & -2 \\ 0 & 4 & 7 \end{vmatrix}$

   (b) $\begin{vmatrix} 1 & 2 & 0 & 9 \\ 2 & 3 & 4 & 6 \\ 1 & 6 & 0 & -1 \\ 0 & -5 & 0 & -8 \end{vmatrix}$

9. Find the inverse of the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
10. Express the quadratic form as a matrix product involving a symmetric coefficient matrix.

    (a) $Q = 8x_1x_2 - x_1^2 - 31x_2^2$

    (b) $Q = 3x_1^2 - 2x_1x_2 + 4x_1x_3 + 5x_2^2 + 4x_3^2 - 2x_2x_3$

11. Prove that for a 3×3 matrix, one may find the determinant by a cofactor expansion along any row or column in the matrix.
12. List all the principle minors of the 4×4 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

13. Prove that:

    (a) Every diagonal matrix whose diagonal elements are all positive is positive definite.

    (b) Every diagonal matrix whose diagonal elements are all negative is negative definite.

    (c) Every diagonal matrix whose diagonal elements are all positive or zero is positive semidefinite.

    (d) Every diagonal matrix whose diagonal elements are all negative or zero is negative semidefinite.

    (e) All other diagonal matrices are indefinite.

14. Determine the definiteness of the following matrices:

    (a) $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

    (b) $\begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix}$

    (c) $\begin{pmatrix} -3 & 4 \\ 4 & -6 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$

(f) $\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 5 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & 6 \end{pmatrix}$

15. Determine the ranks of the matrices in problem 14. How many linearly independent rows are in each?

16. Calculate the determinants of the matrices in problem 14. Which have inverses?