Math Camp Notes: Linear Algebra II

Eigenvalues

Let A be a square matrix. An **eigenvalue** is a number λ which when subtracted from the diagonal elements of the matrix A creates a singular matrix. In other words, and eigenvalue is a number λ such that

$$(A - \lambda I)\,\mathbf{x} = 0, \;\; \mathbf{x} \neq 0$$

Notice also that this implies that an eigenvalue is a number λ such that $A\mathbf{x} = \lambda\mathbf{x}$.
Example:

Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}.$$

Solution: Assume that λ is an eigenvalue of A. Remember that if the matrix resulting when we subtract λI from A is singular, then its determinant must be zero. Then λ solves the equation

$$\left| \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -17 & 8-\lambda \end{pmatrix} \right| = 0$$

$$-\lambda^3 + 8\lambda^2 - 17\lambda + 4 = 0$$

$$-(\lambda - 4)(\lambda^2 - 4\lambda + 1) = 0$$

So one eigenvalue is $\lambda = 4$. To solve the quadratic, we use the quadratic formula:

$$\lambda = 4 \pm \frac{\sqrt{(-4)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = 2 \pm \sqrt{3}$$

Therefore, the eigenvalues are $\lambda = \{2 - \sqrt{3}, 2 + \sqrt{3}, 4\}$.

Eigenvectors

Given an eigenvalue λ, and **eigenvector** is a non-zero vector $\mathbf{x}$ such that $(A - \lambda I)\,\mathbf{x} = 0$.

Example: Find the eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$

We must first find the eigenvalues of the matrix. The determinant of $(A - \lambda I)$ is

$$-\lambda\,(2-\lambda)\,(3-\lambda) + 2\,(2-\lambda) = -\lambda\left(6 - 5\lambda + \lambda^2\right) + 4 - 2\lambda = -8\lambda + 5\lambda^2 - \lambda^3 + 4$$

Therefore, we must find

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda - 1)\,(\lambda - 2)\,(\lambda - 2) = 0.$$

We can see that $\lambda = 1$ and $\lambda = 2$, where $\lambda = 2$ is a repeated root.

Since each eigenvector corresponds to an eigenvalue, let us consider the eigenvalue $\lambda = 1$. The matrix $A - \lambda I$ is then

$$A - \lambda I = \begin{pmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}.$$

This implies that the system of equations can be written

$$x_1 = -2x_2$$

$$x_1 = -x_2 - x_3$$

$$x_1 = -2x_2$$

when $A - \lambda I = \mathbf{0}$. Combining the second and third equation, we have

$$-2x_2 = -x_2 - x_3 \Rightarrow x_2 = x_3$$

Therefore, we have

$$x_1 = -2x_2$$

$$x_2 = x_2$$

$$x_3 = x_2$$

Which implies

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} x_2.$$

Therefore, we can let x_2 be anything we want (say $x_2 = s \in \mathbb{R}$). As we increase and decrease s, we trace out a line in 3-space, which line is the eigenvector of $\lambda = 1$. Each point along that vector has $\det(A - \lambda I) = 0$.

As a check, we can see whether $(A - \lambda I)\,\mathbf{x} = 0$. We then have

$$\begin{pmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2+0-2 \\ -2+1+1 \\ -2+0+2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}.$$

We now know one eigenvector is $\begin{pmatrix} -2 & 1 & 1 \end{pmatrix}'$. We can find the others in a similar manner.

Let $\lambda = 2$. Then

$$(A - \lambda I) = \begin{pmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

which implies

$$x_1 = -x_3,$$

$$x_1 = -x_3,$$

$$x_1 = -x_3.$$

Therefore, we can write the system as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ ??? \\ -1 \end{pmatrix} x_1.$$

Notice that we cannot write x_2 as a function of either x_1 or x_2! Therefore, there is no single eigenvector which corresponds to $\lambda = 2$. However, notice that since the system is independant of x_2 when $\lambda = 2$, we can let x_2 be anything we want. Say $x_1 = s$ and $x_2 = t$. Then we can write the solution to the system as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} s + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t.$$

Therefore, the corresponding eigenvectors are $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. To check that this is the case, notice

$$(A - \lambda I)\,\mathbf{x} = \begin{pmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2+0+2 \\ 1+0-1 \\ 1+0-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}$$

and

$$(A - \lambda I)\,\mathbf{x} = \begin{pmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0+0+0 \\ 0+0+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}.$$

Generally, if the matrix is an $n \times n$ matrix, we have n eigenvalues and n eigenvectors. This is because there are n roots to an nth order polynomial. Also, if an eigenvalue is repeated k times, then there are k corresponding eigenvectors to the repeated root. Also, if a matrix A is only invertible iff $\lambda = 0$ is not an eigenvalue of A.

Diagonalization

A square matrix A is called **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. We state without proof the following theorem:

Let A be $n \times n$, and assume A has n linearly independent eigenvectors. Define the matrices

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}, \quad P = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \dots & \mathbf{p}_3 \end{pmatrix},$$

where $\mathbf{p}_i$ is the ith linearly independant eigenvector. Then $\Lambda = P^{-1}AP$.

Example:

Consider again the matrix

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix},$$

and remember that the corresponding eigenvalues were $\lambda = \{1, 2, 2\}$ and $\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

Therefore, the matrix P can be expressed as

$$P = \begin{pmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Notice that

$$P^{-1} = \begin{pmatrix} -1 & 0 & -1 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then we can see that

$$P^{-1}AP = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} -1 & 0 & -1 \\ 2 & 0 & 4 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Markov Chains

Assume k states of the world, which we label $1, 2, \ldots, k$. Let the probability that we move to state j from state i p_{ij}, and call it the **transition probability**. Then the **transition probability matrix** of the Markov chain is the matrix

$$P = \begin{pmatrix} p_{11} & \cdots & p_{1k} \\ \vdots & \ddots & \vdots \\ p_{k1} & \cdots & p_{kk} \end{pmatrix}.$$

Notice that the rows are the current state, and the columns are the states into which we can move. Also notice that the rows of the transition probability matrix must add up to one (otherwise the system would have positive probability of moving into an undefinied state).

Example:

Say that there are three states of the world: rainy, overcast, and sunny. Let state 1 be rainy, 2 be overcast, and 3 be sunny. Consider the transition probability matrix

$$P = \begin{pmatrix} .8 & .1 & .1 \\ .3 & .2 & .5 \\ .2 & .6 & .2 \end{pmatrix}.$$

For example, if it is rainy, the probability it remains rainy is .8. Also, if it is sunny, the probability that it becomes overcast is .6.

The transition probability matrix tells us the likelihood of the state of the system next period. However, what if we want to know the likelihood of the state in two periods? Call this probability transition matrix P_2. It can be shown that $PP = P_2$. Similarly, it can be shown that

$$P_n = P^n.$$

Example:

If it is rainy today, what is the probability that is will be cloudy the day after tomorrow?

$$P_2 = \begin{pmatrix} .8 & .1 & .1 \\ .3 & .2 & .5 \\ .2 & .6 & .2 \end{pmatrix}\begin{pmatrix} .8 & .1 & .1 \\ .3 & .2 & .5 \\ .2 & .6 & .2 \end{pmatrix} = \begin{pmatrix} .69 & .16 & .15 \\ .40 & .37 & .23 \\ .38 & .26 & .36 \end{pmatrix}$$

Therefore, there is a 0.16 chance that if it is rainy today, then it will be cloudy the day after tomorrow.

Implicitly we have been assuming that we know the vector of probabilities today. For example, we say "assume it is cloudy". But what if we don't know what the probability will be today? We can define an initial state of probabilities as $\mathbf{x} = (p_1, p_2, \ldots, p_n)$. Then the probability that it will be rainy tomorrow is $\mathbf{x}_0 P = \mathbf{x}_1$.

Example:

If there is a 30% chance of rain today, and a 10% chance of sun, what is the probability that is will be cloudy the day after tomorrow?

$$\mathbf{x}_2 = \mathbf{x}_1 P = \mathbf{x}_0 P^2 = \begin{pmatrix} .3 & .6 & .1 \end{pmatrix}\begin{pmatrix} .8 & .1 & .1 \\ .3 & .2 & .5 \\ .2 & .6 & .2 \end{pmatrix}\begin{pmatrix} .8 & .1 & .1 \\ .3 & .2 & .5 \\ .2 & .6 & .2 \end{pmatrix} =$$

$$= \begin{pmatrix} .3 & .6 & .1 \end{pmatrix}\begin{pmatrix} .69 & .16 & .15 \\ .40 & .37 & .23 \\ .38 & .26 & .36 \end{pmatrix} = \begin{pmatrix} .485 & .296 & .219 \end{pmatrix}$$

Therefore, there is a 0.296 chance that it will be cloudy the day after tomorrow.

The **steady-state** vector q of a regular transition matrix P is the unique probability vector that satisfies the equation $\mathbf{q}P = \mathbf{q}$. Notice that this corresponds to $\lambda = 1$ as an eigenvector of P'. It can be shown that $\mathbf{x}P^n \to q$ for all $\mathbf{x}$. Therefore, if we wait long enough in the system, the probabilities of certain events occurring will converge to $\mathbf{q}$.

Example:

What is the probability it will be cloudy as $n \to \infty$?

In order to find the long-run probabilities, we must realize that

$$\mathbf{q}P = \mathbf{q} \Rightarrow P'\mathbf{q}' = \mathbf{q}'.$$

Notice that if $\lambda = 1$, then we have

$$\mathbf{q}P = \lambda\mathbf{q} \Rightarrow P'\mathbf{q}' = \lambda\mathbf{q}'.$$

Therefore, $\mathbf{q}'$ is the eigenvector of P' which corresponds to the eigenvalue of $\lambda = 1$. Therefore, it suffices to find this eigenvalue.

$$(P' - \lambda I)\mathbf{q}' = \begin{pmatrix} -.2 & .3 & .2 \\ .1 & -.8 & .6 \\ .1 & .5 & -.8 \end{pmatrix} \mathbf{q}' = 0 \Rightarrow$$

$$\left(\begin{array}{ccc|c} -.2 & .3 & .2 & 0 \\ .1 & -.8 & .6 & 0 \\ .1 & .5 & -.8 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -\frac{34}{13} & 0 \\ 0 & 1 & -\frac{14}{13} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$q_1 = \frac{34}{13}q_3$$

$$q_2 = \frac{14}{13}q_3$$

$$q_3 = q_3.$$

Therefore, $\mathbf{q} = \left(\frac{34}{13}, \frac{14}{13}, 1\right) q_3$, where q_3 can be whatever we want. Since we must have that the sum of the elements of $\mathbf{q}$ be equal to 1, we choose q_3 accordingly.

$$\frac{34}{13} + \frac{14}{13} + \frac{13}{13} = \frac{61}{13} \Rightarrow q_3 = \frac{13}{61} \Rightarrow \mathbf{q} = \left(\frac{34}{61}, \frac{14}{61}, \frac{13}{61}\right).$$

Homework

1. Find the eigenvalues and the corresponding eigenvectors for the following matrices:

   (a) $\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

   (b) $\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$

   (c) $\begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix}$

   (d) $\begin{pmatrix} -2 & -7 \\ 1 & 2 \end{pmatrix}$

   (e) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

   (f) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2. Determine whether the following matrices are diagonalizable

   (a) $\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$

   (b) $\begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix}$

   (c) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

   (d) $\begin{pmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & -13 & -1 \end{pmatrix}$

   (e) $\begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

3. Diagonalize the following matrices, if possible

   (a) $\begin{pmatrix} -14 & 12 \\ -20 & 17 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

   (d) $\begin{pmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$