Math Camp Notes: Sequences

Definitions

Sequence

A sequence of numbers is an assignment of numbers to the natural numbers. A sequence is denoted as a set with elements labeled from one to n, where n can be either finite or infinite:

$$\{x_n\}_{i=0}^{\infty} = \{x_1, x_2, \ldots, x_n\}$$

Examples:

$$\{x_n\}_{i=0}^{\infty} = \{1, 1, 2, 3, 5, 8, \ldots\}$$
$$\{x_n\}_{i=0}^{\infty} = \{1, 0, 1, 0, \ldots\}$$
$$\{x_n\}_{i=0}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$$
$$\{x_n\}_{i=0}^{\infty} = \{2, 2i, -2, -2i, 2, \ldots\}$$

Sequences can also be defined as functions of n.

Examples:

$$a(n) = n^2 \Rightarrow \{x_n\}_{i=0}^{\infty} = \{0, 1, 4, 16, \ldots\}$$
$$b(n) = \frac{n}{n+1} \Rightarrow \{x_n\}_{i=0}^{\infty} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$$
$$c(n) = \sqrt{\ln(n)} \Rightarrow \{x_n\}_{i=1}^{\infty} = \left\{\sqrt{\ln(1)}, \sqrt{\ln(2)}, \ldots\right\}$$
$$d(n) = \frac{e^n}{n} \Rightarrow \{x_n\}_{i=1}^{\infty} = \left\{e, \frac{e^2}{2}, \frac{e^3}{3}, \ldots\right\}$$

Boundedness

We say a sequence is bounded above if there exists a number N_a such that all elements of the sequence are less than N_a, and we call N_a an upper bound. We say a sequence is bounded below if there exists a number N_b such that all elements of the sequence are greater than N_b, and we call N_b a lower bound. A sequence is bounded if it is bounded both above and below. Furthermore, the smallest number N_a which is an upper bound of the sequence is called the least upper bound, while the largest number N_b which is a lower bound of the sequence is called the lowest upper bound. Upper and lower bounds need not exist.

Example:

The sequence

$$\{x_n\}_{i=1}^{\infty} = \{0, 1, 0, 2, 0, 3, 0, 4, \ldots\}$$

has zero as its greatest lower bound and has no least upper bound (or any upper bound, for that matter). Therefore, the sequence is bounded below, but is not bounded since it is not bounded above.

Increasing and Decreasing

A sequence is:

1. (Weakly) Increasing or non-decreasing iff $x_n \leq x_{n+1}$.
2. Strictly increasing iff $x_n < x_{n+1}$.
3. (Weakly) Decreasing or non-increasing iff $x_n \geq x_{n+1}$.
4. Strictly decreasing iff $x_n > x_{n+1}$.

Limits, Convergence, and Divergence

A sequence $\{x_n\}_{i=1}^{\infty}$ has a limit point $L \in \mathbb{R}$ iff for each $\epsilon > 0$, $\exists\ K \in \mathbb{Z}^{++}$ such that if $n \geq K$, then $|x_n - L| < \epsilon$. In other words, there must be a number K such that all elements after the Kth element must be in the epsilon ball $I_\epsilon(L)$. We say a sequence converges if it has a limit point. If it has no limit point, then we say it diverges.

Example:

Prove the sequence $\{x_n\}_{i=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ converges.

Proof: It suffices to show the sequence has a limit point. Consider $L = 0$. We must show that for each $\epsilon > 0$, $\exists\ K \in \mathbb{Z}^{++}$ such that if $n \geq K$, then $|x_n| < \epsilon$.

$$|x_n| < \epsilon \Rightarrow |\frac{1}{n}| < \epsilon \Rightarrow \frac{1}{n} < \epsilon \Rightarrow 1 < n\epsilon \Rightarrow n > \frac{1}{\epsilon}.$$

Choose $\epsilon > 0$. Then any integer $K > \frac{1}{\epsilon}$ works.■ To see that $K > \frac{1}{\epsilon}$ actually works, just let ϵ different numbers. For example, let $\epsilon = 1$. Then our chosen K must be greater than 1. From our sequence, we can easily see that any number $\frac{1}{n}$ where $n > 1$ is going to be less than $\epsilon = 1$. If $\epsilon = \frac{1}{2}$, then our K must be greater than 2. We can see that for every $n > 2$, $\frac{1}{n} < \epsilon = \frac{1}{2}$.

For spaces of dimension greater than $\mathbb{R}$, we no longer use an interval to define an epsilon ball. Instead, we must use a different measure of distance called the norm. The norm (or distance) between two points is denoted by the function $d(x, y) = ||x - y||$. One can use all sorts of definitions for what the "distance" actually means. The most commonly used distance is the Euclidean distance, which is defined in $\mathbb{R}^N$ as

$$d(x, y) = ||x - y|| = \sqrt{\sum_{i=1}^{N}(x_i - y_i)^2}.$$

Other distances could be street or grid distances. For example if you are on the south-west corner of a city block and you want to go to the north-east corner of the same block, you must travel east one block and north one block. The grid distance you walked is two blocks, whereas the Eucildean distance is $\sqrt{2}$ blocks.

The epsilon ball about r for dimensions greater than $\mathbb{R}^N$ is defined as $B_\epsilon(r) = \{x \in \mathbb{R}^N :\ d(x, r) < \epsilon\}$. It can be shown that a sequence of vectors in $\mathbb{R}^N$ converges iff all the component sequences converge in $\mathbb{R}$.

Properties

Absolute values in $\mathbb{R}$ have the following properties, both forms of the triangle inequality:

1. $|x + y| \leq |x| + |y|$
2. $||x| + |y|| \leq |x - y|\ \forall\ x, y \in \mathbb{R}$

The following is a form of the triangle inequality for norms:

1. $d(x, y) \leq d(x, z) + d(z, y)$

Homework

Find the greatest lower bound and the least upper bound of the following sequences. Also, prove whether they are convergent or divergent:

1. $\{x_n\}_{i=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$
2. $\{x_n\}_{i=1}^{\infty} = \{-1, 1, -1, 1, \ldots\}$
3. $\{x_n\}_{i=1}^{\infty} = \left\{-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \ldots\right\}$

Prove the following:

1. A sequence can only have at most one limit.
2. If $\{x_n\}_{i=1}^{\infty} \to x$ and $\{y_n\}_{i=1}^{\infty} \to y$, then $\{x_n + y_n\}_{i=1}^{\infty} = x + y$.
3. A sequence of vectors in $\mathbb{R}^N$ converges iff all the component sequences converge in $\mathbb{R}$.
4. If $\{x_n\}_{n=1}^{\infty} \to x$, and $\{y_n\}_{n=1}^{\infty} \to y$, then $\{x_n + y_n\}_{n=1}^{\infty} \to x + y$. (Do in class if have extra time)
5. The sequence $\{x_n\}_{n=1}^{\infty} = \left\{(1, \frac{1}{2}), (1, \frac{1}{3}), (1, \frac{1}{4}), \ldots\right\}$ converges to $(1, 0)$.
6. The sequence $\{x_n\}_{n=1}^{\infty} = \left\{(\frac{1}{2}, \frac{1}{2}), (\frac{2}{3}, \frac{1}{3}), (\frac{3}{4}, \frac{1}{4}), \ldots\right\}$ converges to $(1, 0)$.