Math Camp Notes: Sets and Functions

Sets

The following are useful definitions:

1. **Open Ball:** An open ball $B(x, r)$ in $\mathbb{R}^N$ with center x and radius r is the set $B(x, r) = \{y \in \mathbb{R}^N : d(x, y) < r\}$. It is the same as the epsilon ball we defined in $\mathbb{R}^N$ previously.

2. **Closed Ball:** A closed ball $\bar{B}(x, r)$ in $\mathbb{R}^N$ with center x and radius r is the set $B(x, r) = \{y \in \mathbb{R}^N : d(x, y) \leq r\}$.

3. **Open Set:** A set $X \in \mathbb{R}^N$ is open if $\forall\ x \in X,\ \exists\ r > 0$ such that $B(x, r) \subset X$.

4. **Closed Set:** A set $X \in \mathbb{R}^N$ is closed if its compliment is open. An equivalent definition is that a set is closed iff $\forall$ sequences $\{x_k\}$ with $x_k \in X\ \forall\ k$ and $\{x_k\} \to x$, then $x \in X$.

5. **Bounded Set:** A set $X \in \mathbb{R}^N$ is bounded if $\exists\ r > 0$ such that $B(0, r)$.

6. **Compact Set:** A set $X \in \mathbb{R}^N$ is compact if $\exists$ a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ and a point $x \in X$ such that $x_{k_j} \to x$. In other words, a set is compact if every sequence has a convergent subsequence. This is equivalent to being closed and bounded.

7. **Convex Combination:** Given any finite number of points $\{x_1, \ldots x_n\}$, $x_i \in \mathbb{R}^N$, a point $z \in \mathbb{R}^N$ is a convex combination of the points $\{x_1, \ldots x_n\}$ if $\exists\ \lambda \in \mathbb{R}^N_+$ satisfying $\sum_{i=1}^N \lambda_i = 1$ such that $z = \sum_{i=1}^N \lambda_i x_i$.

8. **Convex Set:** A set is convex if the convex combination of any two points in the set is also contained in the set. For example, the unit disc is a convex set in $\mathbb{R}^2$, while the unit circle is not. Another way of stating this is if $x_1, x_2 \in X$, then $\alpha x_1 + (1 - \alpha) x_2 \in X$, where $\alpha \in [0, 1]$.

Functions

Definitions

1. **Function:** A function F from a set A to a set B is a rule that assigns to each element of A one and only one object in B. It is denoted $F : A \to B$.

2. **Domain:** In the above example, A is the domain.

3. **Target Space:** In the above example, B is the target space.

4. **Image or Range:** $F(x) \in B$ is the element of B which F assigns to the element $x \in A$. We say $y = F(x)$ is the image of x under F. The set of all $F(x)$ for $x \in A$ is called the image of A. For a set $C \subset A$, the image of C under F is $F(C) = \{b \in B : b = F(a)$ for some $a \in C\}$.

5. **Pre-image:** The pre-image of V under F is the set of all points in the domain whose image lies in V, or $F^{-1}(V) = \{a \in A : F(a) \in V\}$. Notice that the image of C is not necessarily the image of the preimage of the image of C. However, the image of the pre-image of V is always V. For example, let $F(x) = 1$, and let $U = \{2\}$. Then the image of U is $\{1\}$, the pre-image of $\{1\}$ is $\mathbb{R}$, whose image is (1). So the image of the pre-image of $\{1\}$ is $\{1\}$, and the image of the preimage of the image of U is $\{1\} \neq U$.

6. **Injective or one-to-one:** A function is injective or one-to-one if $\forall\ b \in F(A)$, $\exists$ only one $x \in A$ with $y = f(x)$. In words, a function is one-to-one if every output of the function has at most one input.

7. **Surjective:** If for each element $b \in B\ \exists\ a \in A$ such that $b = F(a)$, then the function is surjective. In other words, the image of F under A is equal to the whole target space.

8. **Inverse:** If a function F is both injective and surjective, then it has an inverse function $F^{-1}(y)$ such that $x = F^{-1}[F(x)]$ for all x in the domain.

9. **Monotonicity:** A function is monotonic if it is either non-increasing or non-decreasing. It is strictly monotonic if it is either strictly increasing or strictly decreasing.

10. **Continuity:** Consider a function $F : X \to Y$. If for any sequence $\{x_n\} \subset X$ such that $\{x_n\} \to x \in X$, the corresponding sequence $\{y_n = F(x_n)\} \to y = F(x) \in Y$, then F is continous at $x \in X$.

11. **Continuous differentiability:** A function $F : \mathbb{R}^N \to \mathbb{R}$ is continuously differentiable if it is differentiable and its derivative function is continuous. This means that its gradient exists and is continuous (i.e. that each partial derivative is a continous function in $\mathbb{R}$ as defined the first day of class). The set of all continuously differentiable functions is called C^1. If a function's total derivative is itself continuously differentiable (each element of the Hessian matrix is a continously differentiable function in $\mathbb{R}$), then we say it is twice differentiable. We denote the set of all twice differentiable functions as C^2.

12. **Boundedness:** A function $F : \mathbb{R}^N \to \mathbb{R}$ is bounded if $\exists\ K \in \mathbb{R}$ such that $|F(x)| < K\ \forall\ x \in X$. A function $F : \mathbb{R}^N \to \mathbb{R}^M$ is bounded if $\exists\ K \in \mathbb{R}$ such that $d(F(x), 0) < K\ \forall\ x \in X$.

13. **Concavity:** A function $F : \mathbb{R} \to \mathbb{R}$ is concave if $\forall\ \alpha \in (0, 1)$ and $x_1, x_2 \in \mathbb{R}$, $F(\alpha x_1 + (1 - \alpha)x_2) \geq \alpha F(x_1) + (1 - \alpha)F(x_2)$. Graphically, this is a function where the chord drawn between any two points on its graph lies completely below or on the graph.

14. **Convexity:** A function $F : \mathbb{R} \to \mathbb{R}$ is convex if $\forall\ \alpha \in (0, 1)$ and $x_1, x_2 \in \mathbb{R}$, $F(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha F(x_1) + (1 - \alpha)F(x_2)$. Graphically, this is a function where the chord drawn between any two points on its graph lies completely above or on the graph. This is not to be confused with a convex set.

Weierstrauss Theorem

The Weierstrauss Theorem is one of the most important in economics. It states conditions under which we are guaranteed to find a solution to a maximization problem. Since economics is all about maximizing things such as utility and profits, we state it here without proof.

Weierstrauss Theorem: Let $D \subset \mathbb{R}^N$ be a compact set, and $f(x) : D \to \mathbb{R}$ a continuous function. Then f attains a maximum and a minimum on D, i.e. $\exists\ z_1, z_2 \in D$ such that $f(z_1) \geq f(x) \geq f(z_2)\ \forall\ x \in D$.

Homework

State whether the following sets are open, closed, neither, or both:

1. $\{(x, y) : -1 < x < 1, y = 0\}$
2. $\{(x, y) : x, y \text{ are integers}\}$
3. $\{(x, y) : x + y = 1\}$
4. $\{(x, y) : x + y < 1\}$
5. $\{(x, y) : x = 0 \text{ or } y = 0\}$

Prove the following:

1. Open balls are open sets
2. Any union of open sets is open
3. The finite intersection of open sets is open

4. Any intersection of closed sets is closed
5. The finite union of closed sets is closed
6. All strictly monotonic continuous functions $F : \mathbb{R} \to \mathbb{R}$ have an inverse.
7. Let f and g be functions from $\mathbb{R}^k$ to $\mathbb{R}^m$ which are continuous at x. Then $h = f - g$ is continuous at x.
8. Let f and g be functions from $\mathbb{R}^k$ to $\mathbb{R}^m$ which are continuous at x. Then $h = fg$ is continuous at x.

For each function, determine whether the it definitely has a maximum, definititely does not have a maximum, or that there is not enough information to tell, using the Weierstrauss Theorem. If it definitely has a maximum, prove that this is the case.

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x.$
2. $f : [-1, 1] \to \mathbb{R},\ f(x) = x$
3. $f : (-1, 1) \to \mathbb{R},\ f(x) = x$
4. $f : [-1, 1] \to \mathbb{R},\ f(x) = \begin{cases} 0 \text{ if } x = 1 \\ x \text{ otherwise} \end{cases}$
5. $f : \mathbb{R}_{++} \to \mathbb{R},\ f(x) = \begin{cases} 1 \text{ if } x = 5 \\ 0 \text{ otherwise} \end{cases}$

Consider the standard utility maximization problem

$$\max_{x \in B(p,I)} U(x), \text{ where } B(p, I) = \{x \in \mathbb{R}^n_+ |\ p \cdot x \leq I\}$$

Prove a solution exists for any $U(x)$ continuous, $I > 0$ and $p \in \mathbb{R}^n_{++}$. Show a solution may not necessarily exist if $p \in \mathbb{R}^n_+$.