Zero-point energy of a cylindrical layer of finite thickness

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The zero-point electromagnetic (Casimir) energy of an infinitely long cylindrical layer of finite thickness is calculated. The dielectric and magnetic properties of the layer and those of the surrounding medium are assumed to be described by real constant dielectric permittivities and magnetic permeabilities satisfying the relationship of constant speed of light across the interfaces. We use the mode summation technique together with the Riemann ζ-function regularization procedure to remove the occurring divergences from the Casimir energy. We present analytical expressions and numerical calculations for various limiting cases in terms of the radial dimension, curvature, and material composition of the cylindrical layer.

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I. INTRODUCTION

Long-range dispersion forces, such as van der Waals forces (between a pair of unpolarized atoms or molecules), Casimir-Polder forces (between an atom and a macroscopic object), and Casimir forces (between two macroscopic objects), originate from the vacuum fluctuations of the electromagnetic field [1–6]. They couple electrically neutral objects with no permanent electric and/or magnetic moments and are quantum mechanical in their physical nature. These long-range forces become especially important for systems integrated in nanotechnological devices, where deep understanding of the behavior of various miniaturized components plays a crucial role [7,8].

Processes, such as friction, adhesion, and wear, are directly related to the Casimir forces and they can be dominant at small scales. Thus qualitative and quantitative knowledge is necessary in order to be able to control or avoid unwanted effects [9]. Also, many nanostructured materials offer the possibility for fundamental studies of Casimir forces in non-trivial geometries, as well as their manifestation in practical devices. For example, multiwall carbon nanotubes are quasi-one-dimensional structures consisting of concentric cylindrically wrapped graphene sheets, and their stability originates from such long-ranged forces [10,11]. In addition, oscillating carbon nanotubes or buckyballs inside a stationary carbon nanotube due to these long range forces have been demonstrated recently [12,13].

Considerable progress in improving the precision of experiments has led to many studies in measuring the Casimir force in various systems. For example, using a torsional pendulum the force between metallic surfaces was measured with high accuracy [14]. Measurements using an atomic force microscope have also been reported for the force between metallic disks and spheres [15–17]. In addition, experiments for measuring the force between metallic or dielectric parallel plates [18], dielectric spheres and plates [19,20], metallic or dielectric cylinder and a plane [21], and dielectric crossed cylinders [22,23] have been published. Furthermore, advanced experimental techniques to measure the Casimir forces in the near-surface regime (nm range) [24] or even quadrupole corrections to the Casimir interaction [25] have been reported.

These achievements, together with the potential importance of the Casimir effect in nanostructured devices, have spurred many theoretical investigations using a variety of methods for systems with different geometries and materials [2]. Recently, calculations for the Casimir self-energy of a dielectric ball [26], a metallic ball [27], a solid dielectric cylinder [3,5], a parallelepiped [28], a cylinder and a plane [21], and two conducting eccentric cylinders [4] have been reported also. It appears that the magnitude and sign of the Casimir force depend strongly on the geometry, topology, and/or the types of materials of the system under consideration. For many systems the force is found to be attractive [14,18,29,30], but for a dielectric ball it is calculated to be repulsive [26], for a rectangular box it changes sign as a function of the dimensions of the box [31,32], and for a dielectric cylinder it was calculated to be zero [3,5]. Interestingly the force is repulsive for anisotropic and permeable rectangular plates [33], and it can be repulsive or attractive for media of excited atoms depending on the resonance frequency [34].

There are a number of methods used to calculate the Casimir force in nonplanar geometries. One of the most advanced approaches is based on macroscopic quantum electrodynamics of dispersing and absorbing media (see Ref. [1] for latest review). There, the vacuum electric and magnetic fields are considered as primary physical observables whose quantum mechanical operators are given by the convolution of the Green’s tensor of the Fourier-domain Maxwell equations with appropriately chosen bosonic field operators for creation and annihilation of the medium-assisted electromagnetic field quanta. In this approach, the Green’s tensor takes into account the geometry of the system, while the (complex) dielectric and magnetic permeabilities are responsible for the dissipation. The method was successfully used to calculate the long-range dispersion forces in carbon nanotubes [35,36] and nonempty planar cavities [37]. Other methods, such as the proximity-force approximation [38], semiclassical approximation [4,39], optical approximation [40], and path integral methods [41] have been reported.

The mode summation is another approach that is of particular interest. It leads to the final result for the Casimir energy in an elegant way in cases where the effects of medium absorption are less important and may be neglected.
The mode summation technique has been applied to systems with circular [42], spherical [27, 43, 44], and cylindrical [3–6] geometries. It involves the use of contour integration in the complex frequency plane [45], where two sums appear in the Casimir energy expression. One is over the roots of the dispersion equation at a fixed value of the angular moment, the other one is over the angular moment, and both series are divergent. The divergency from the first sum is removed by subtracting off the vacuum energy of an infinite homogeneous space. The divergency from the second sum is removed by making use of different regularization techniques. For example, results have been reported for Casimir energy calculations using the Riemann \( \zeta \)-function [3, 5, 45–48] and exponential cutoff [4, 6, 43, 44] techniques.

In this work, we extend the application of the mode summation technique to the calculation of the Casimir energy of a cylindrical layer with a finite thickness imbedded in some medium. The permittivity \( \varepsilon \) and permeability \( \mu \) for the layer, and the permittivity \( \varepsilon_m \) and permeability \( \mu_m \) for the medium are taken to be constants. To account for the dielectric properties in the Casimir energy turns out to be difficult for arbitrary dispersionless permittivity and permeability. This situation also occurs when calculating the Casimir effect for a ball [5, 26] or a cylinder [3, 5], where the condition of constant light velocity across the interface is imposed. Here we also impose the same condition \( \varepsilon \mu = \varepsilon_m \mu_m = c^2 \), where \( c \) is the speed of light. It turns out that if the parameter \( \xi = \frac{\varepsilon \mu}{\varepsilon_m \mu_m} \) is introduced, the expression for the Casimir energy of the cylindrical layer can be represented as an infinite series in terms of powers of \( \xi^2 \) in analogy to the calculations of the Casimir energy for a solid ball or cylinder [3, 5]. The \( \xi^2 \) expansion allows one to consider explicitly each term in the series, and thus establish various limiting cases. We focus on the case \( \xi \ll 1 \) which corresponds to an optically dilute dielectric-diamagnetic cylindrical layer with properties not very different from the dielectric and magnetic properties of the medium, and on the case \( \xi = 1 \) which corresponds to two perfectly conducting concentric cylindrical shells. Also, here we use the Riemann \( \zeta \)-function regularization procedure to remove the occurring divergences in the considered \( \xi^2 \) terms, thus extending its application to cylindrical layer systems. These simplified situations and the obtained different functionalities with respect to the dimensions and curvature of the system can serve as a limiting case and a test for a future theory which may include more realistic dielectric and magnetic characteristics of the material and the medium.

The paper is organized as follows. In Sec. II we show the results for the electromagnetic modes for the dielectric layer under the condition \( \varepsilon \mu = \text{const} \). In Sec. III we present the calculations for the Casimir energy for the \( \xi \ll 1 \) and \( \xi = 1 \) limits using the mode summation method and the Riemann \( \zeta \)-function regularization procedure for removing the unphysical divergences. In Sec. IV we give the conclusions.

II. ELECTROMAGNETIC MÖDES UNDER THE CONDITION \( \varepsilon \mu = \text{const} \)

The system under consideration is a cylindrical layer with an inner radius \( R_1 \) and outer radius \( R_2 \) with an infinite axial direction. The dielectric layer has dielectric and magnetic functions \( \varepsilon \) and \( \mu \), respectively, and it is placed in an infinite medium of dielectric and magnetic functions \( \varepsilon_m \) and \( \mu_m \)—see Fig. 1. In order to utilize the mode summation method, one needs to find the electromagnetic modes by solving the Maxwell’s equations [49] with the appropriate boundary conditions across interfaces I and II.

The electric and magnetic fields are expressed in cylindrical coordinates \((\rho, \theta, z)\) as follows [49]:

\[
E^e_\rho = \sum_{n=-\infty}^{\infty} \left( \frac{n \omega}{\rho X} \left[ C_n^J J_n^J + D_n^H H_n^H \right] - \frac{ik_z}{\lambda} \left[ A_n^J J_n^J + B_n^H H_n^H \right] \right) \times e^{i\theta} e^{i(k_z z - \omega t)},
\]

\[
E^e_\theta = \sum_{n=-\infty}^{\infty} \left( nk_z^2 \left[ A_n^J J_n^J + B_n^H H_n^H \right] + i\omega \left[ C_n^J J_n^J + D_n^H H_n^H \right] \right) \times e^{i\theta} e^{i(k_z z - \omega t)},
\]

\[
E^e_z = \sum_{n=-\infty}^{\infty} \left( A_n^J J_n^J + B_n^H H_n^H \right) e^{i\theta} e^{i(k_z z - \omega t)},
\]

\[
B^e_\rho = \sum_{n=-\infty}^{\infty} \left( - \frac{n \omega \varepsilon \mu}{\rho X} \left[ A_n^J J_n^J + B_n^H H_n^H \right] - \frac{ik_z}{\lambda} \left[ C_n^J J_n^J + D_n^H H_n^H \right] \right) \times e^{i\theta} e^{i(k_z z - \omega t)},
\]

\[
B^e_\theta = \sum_{n=-\infty}^{\infty} \left( \frac{nk_z}{\rho X} \left[ C_n^J J_n^J + D_n^H H_n^H \right] - \frac{i\varepsilon \omega \mu}{\lambda} \left[ A_n^J J_n^J + B_n^H H_n^H \right] \right) \times e^{i\theta} e^{i(k_z z - \omega t)},
\]

\[
B^e_z = \sum_{n=-\infty}^{\infty} \left( C_n^J J_n^J + D_n^H H_n^H \right) e^{i\theta} e^{i(k_z z - \omega t)},
\]

where \( j = 1, 2, 3 \) stands for the three regions separated by the interfaces I and II in Fig. 1, \( k_z \) is the wave vector along the \( z \) direction, \( \omega \) is the frequency of the electromagnetic excitations, and \( \lambda = \varepsilon \mu \omega^2 - k_z^2 \). Also, \( J_n = J_n(\chi \rho) \) and \( H_n^{(1)} = H_n^{(1)}(\chi \rho) \) are the Bessel function of first kind of order \( n \) and the Hankel functions of the first kind of order \( n \), respectively.

FIG. 1. (Color online) Cylindrical layer of finite thickness with its axial direction perpendicular to the page is immersed in an infinite medium. The permittivity and permeability of the layer are \( (\varepsilon, \mu) \), and those of the medium are \( (\varepsilon_m, \mu_m) \). The interfaces are denoted as I and II.
In addition, \( J_n'(x) = \frac{dJ_n(x)}{dx} \), and \( H_n^{(1)}(x) = \frac{dH_n^{(1)}(x)}{dx} \).

The unknown coefficients \( A_n^1, B_n^1, C_n^1, D_n^1 \) are related by imposing the boundary conditions for the continuity of \( \varepsilon E_x, E_y, E_z, B_z \) across each interface of the cylindrical layer giving the dispersion relation for the electromagnetic modes supported by this system. In general, the dispersion equation for cylindrical structures can be expressed in terms of a determinant of a 4 \times 4 matrix where separation between pure magnetic (TE) and pure electric (TM) modes is not possible except for the \( n=0 \) case \([50,51]\).

Here we impose the condition of a constant speed of light \( c \) across each interface \( \varepsilon \mu = \varepsilon_m \mu_m = c^{-2} \) in order to facilitate the calculation of the Casimir energy. The \( \varepsilon \mu = \varepsilon_m \mu_m = c^{-2} \) condition is referred to as the dielectric-diamagnetic case as opposed to the purely dielectric one in which \( \mu_1 = \mu_2 = 1 \). The physical implication of this condition is that the electric and magnetic fields are treated in a symmetric way, thus there is no preference for the electric field as it occurs for the dielectric cylinder \([52–54]\).

After applying the boundary conditions and the constant speed of light condition, we obtain the following four equations for the TM modes:

\[
e_n A_n^1 J_n^1(\chi R_2) = \varepsilon \left[ A_n^2 J_n^1(\chi R_1) + B_n^2 H_n^{(1)}(\chi R_1) \right],
\]

\[
e_m B_n^1 H_n^{(1)}(\chi R_2) = \varepsilon \left[ A_n^2 J_n^1(\chi R_2) + B_n^2 H_n^{(1)}(\chi R_2) \right],
\]

\[
A_n^2 J_n^1(\chi R_1) = A_n^2 J_n^1(\chi R_1) + B_n^2 H_n^{(1)}(\chi R_1),
\]

\[
B_n^2 H_n^{(1)}(\chi R_2) = A_n^2 J_n^1(\chi R_2) + B_n^2 H_n^{(1)}(\chi R_2),
\]

where for the \( r < R_1 \) region, \( B_n^1 = 0 \), and for the \( r > R_2 \) region, \( A_n^2 = 0 \) due to the properties of the Bessel functions and the requirement of a finite solution for the electric and magnetic field in these regions \([49]\). A similar set of equations can be obtained for the TE modes, where \( A_n^1 \) is substituted by \( C_n^1, B_n^1 \) is substituted by \( D_n^1, \varepsilon_m \) is substituted by \( \mu, \) and \( \varepsilon \) is substituted by \( \mu_m \). Thus we find that the dispersion relations are decoupled and are given by the following expressions:

\[
f_n^{TE}(\chi, R_1, R_2) = \Delta_1(\chi R_1)\Delta_2(\chi R_2) + (\mu - \mu_m)^2 J_n(\chi R_1)J_n^1(\chi R_1) \times H_n^{(1)}(\chi R_2)H_n^{(1)}(\chi R_2) = 0,
\]

\[
f_n^{TM}(\chi, R_1, R_2) = \delta_1(\chi R_1)\delta_2(\chi R_2) + (\varepsilon - \varepsilon_m)^2 J_n(\chi R_1)J_n^1(\chi R_1) \times H_n^{(1)}(\chi R_2)H_n^{(1)}(\chi R_2) = 0,
\]

for each \( n=0, \pm 1, \pm 2, \ldots \). We have made the following notations:

\[
\Delta_1(\chi R_1) = \mu J_n(\chi R_1)H_n^{(1)}(\chi R_1) - \mu_m J_n(\chi R_1)H_n^{(1)}(\chi R_1),
\]

\[
\delta_1(\chi R_1) = \varepsilon J_n(\chi R_1)H_n^{(1)}(\chi R_1) - \varepsilon_m J_n(\chi R_1)H_n^{(1)}(\chi R_1),
\]

where \( r=1,2 \).

### III. Casimir Energy of a Cylindrical Layer

#### A. The mode summation method and the Riemann \( \zeta \)-function

One of the simplest and most intuitive methods to compute the Casimir energy is to express it in terms of a sum of the ground state (zero-point) photon energies. From the dispersion relation \( f_n^{TE}(\chi, R_1, R_2) f_n^{TM}(\chi, R_1, R_2) = 0 \) in Eqs. (11)–(14), one obtains the zero-point photon energy. The Casimir energy is then formally expressed by the sum over all modes, as follows \([3–6]\):

\[
E_C = \frac{\hbar}{2} \sum_{\{p\}} (\omega_p - \tilde{\omega}_p),
\]

where \( \omega_p \) are the eigenfrequencies satisfying Eqs. (11) and (12) and \( \tilde{\omega}_p \) are the ones corresponding to the reference vacuum with no boundaries present, \( R_{1,2} \to \infty \) (Fig. 1). Here \( \{p\} \) are the complete set of quantum numbers determined by the geometry of the system. In the case of a cylindrical structure, \( \{p\} = (n, m, k) \) where \( n \) is the order of the Bessel functions, \( m \) denotes the number of roots of Eqs. (11) and (12) and \( k \) is continuous and corresponds to the wavevector along the infinite axial direction. Therefore, the Casimir energy per unit length of the cylindrical layer becomes

\[
E_C = \frac{\hbar}{2} \int_{-\infty}^{\infty} \frac{dk_z}{\pi m_n^m} \sum_{n,m} \left[ \omega_{n,m}(k_z) - \tilde{\omega}_{n,m}(k_z) \right],
\]

where \( \omega_{n,m}(k_z) \) are the solutions to Eqs. (11) and (12). Since both sums in Eq. (16) are divergent, a means of regularization procedure is required. Researchers have used different ways to remove the divergences. For example, in Refs. \([4,6,43,44]\) an exponential cutoff technique has been applied to calculate the Casimir energy of two coaxial, eccentric cylinders as well as for a conducting sphere and dielectric ball. In Refs. \([3,5,45–48]\), the Riemann \( \zeta \)-function regularization procedure has been used for the Casimir energy of an infinite cylinder. In Ref. \([55]\), the Hurwitz \( \zeta \)-function regularization technique has been used for the Casimir energy of a piecewise uniform, closed string. In Ref. \([56]\), frequency cutoff regularization was taken to calculate the Casimir effect of a conducting sphere.

In this work, we apply the Riemann \( \zeta \)-function technique in a similar fashion as it has been done in Refs. \([3,5,45–48]\). Equation (16) is represented as

\[
E_C = \lim_{\epsilon \to 0} E_C(s),
\]

\[
E_C(s) = \frac{\hbar}{2} \int_{-\infty}^{\infty} \frac{dk_z}{\pi m_n^m} \sum_{n,m} \left[ \omega_{n,m}^{+}(k_z) - \omega_{n,m}^{-}(k_z) \right],
\]

where \( s \) in general is a complex number, allowing one to perform the regularization procedure. Applying the condition \( \varepsilon \mu = \text{const} \) with constant \( \varepsilon \) and \( \mu \) ensures that all divergences between the interior and exterior modes cancel \([45,54,56]\).

Those divergences are regularized by employing the Riemann \( \zeta \)-function regularization technique here. In principle, \( \varepsilon \) and \( \mu \) may be frequency dependent thus including medium dispersion. Recent studies have shown, however, that even for objects made out of dispersive materials, the divergences...
in the Casimir energy still exist, thus requiring a proper regularization procedure [57,58].
Using \( \chi = \epsilon_2 \omega^2 - k_z^2 \), \( E_c(s) \) is expressed as
\[
E_c(s) = \frac{\hbar c^s}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \left[ \chi_{nm}^2 (R_1, R_2) + k_z^2 \right]^{s/2} - \left[ \chi_{nm}^2 (R_1 \to -\infty, R_2 \to +\infty) + k_z^2 \right]^{s/2}.
\]
(18)

Using the residue’s theorem, the infinite sum over \( m \) can be converted in terms of a contour integral in the complex plane for any given value of \( n \) and \( k_z \) [3-5,27,43],
\[
E_c(s) = \frac{\hbar c^{-s}}{4\pi i} \sum_{m=-\infty}^{\infty} \frac{dk}{2\pi} \oint_C d\chi (\chi^2 + k_z^2)^{-s/2} \\
\times d\chi \ln \left[ \frac{f^{TM}(iR_1, \chi R_2) f^{TM}(\chi R_1, iR_2)}{f^{TM}(i\infty) f^{TM}(i\infty)} \right].
\]
(19)

where the contour \( C \) is along the imaginary axis \((-i\infty, +i\infty)\) and an infinite semicircle closed in the right half of the complex plane. \( f^{TM}(i\infty) \) are the dispersion relations when there are no boundaries in the system.

Following the arguments in Refs. [3,5] and making the substitution \( y = \ln \chi \), \( E_c(s) \) is reduced to
\[
E_c(s) = \frac{\hbar c^{-s}}{2\pi} \sin \frac{\pi s}{2} \sum_{m=-\infty}^{\infty} \int_0^{\infty} dy \left( y^2 - k_z^2 \right)^{-s/2} \\
\times d\chi \ln \left[ \frac{f^{TM}(iR_1, iR_2) f^{TM}(iR_1, iR_2)}{f^{TM}(i\infty) f^{TM}(i\infty)} \right].
\]
(20)

Changing the order of integration and then performing the integration over \( k_z \) one obtains [3,5]
\[
E_c(s) = \frac{\hbar c^{-s}}{4\pi i} \frac{1}{s-1} \Gamma \left( \frac{3-s}{2} \right) \\
\times \sum_{m=-\infty}^{\infty} \int_0^{\infty} dy y^{s-1} \frac{d}{dy} \ln \left[ \frac{f^{TM}(iR_1, iR_2) f^{TM}(iR_1, iR_2)}{f^{TM}(i\infty) f^{TM}(i\infty)} \right],
\]
(21)

where \( \Gamma(s) \) is the Gamma function.

\[ \] B. \( \xi = (\epsilon - \epsilon_m) / (\epsilon + \epsilon_m) \ll 1 \) case

We introduce the parameter \( \xi = (\epsilon - \epsilon_m) / (\epsilon + \epsilon_m) \) following the works for compact dielectric-diamagnetic cylinders and spherical balls [3,5,48,59]. The functions \( J_n(iR) \) and \( H_n^{(1)}(iR) \) are replaced by the modified Bessel functions using \( J_n(iy) = i^n J_n(iy) \) and \( K_n(iy) = \pi i^{n+1}/2 H_n^{(1)}(iy) \). Further, using standard properties of the Bessel functions [60], the expression for the dispersion relations is written as
\[
f^{TM}_n(iR_1, iR_2) f^{TM}_n(iR_1, iR_2) = 16\mu^2 \varepsilon_m^2 \pi^4 \frac{1}{\sqrt{y^2 R_1^2 R_2^2}} \left( 1 - \xi^2 \right) \left( 1 + \xi^2 \right) X_1 \\
+ \frac{16\varepsilon^2}{(1 + \xi^2)} X_2,
\]
(22)

where the terms \( X_1 \) and \( X_2 \) are defined as:
\[
X_1 = \frac{2I_n(yR_1)I_n''(yR_1)K_n(yR_2)K_n''(yR_2)}{(y^2 R_1^2 R_2^2)} - \frac{I_n(yR_1)I_n'(yR_1)K_n(yR_2)K_n'(yR_2)}{(y^2 R_1)^2} \\
- \frac{I_n(yR_2)I_n'(yR_2)K_n(yR_2)K_n'(yR_2)}{(y^2 R_2)^2},
\]
(23)

\[
X_2 = \frac{I_n(yR_1)I_n'(yR_1)K_n(yR_2)K_n'(yR_2)}{(y^2 R_1^2 R_2^2)} \\
\times \left[ I_n(yR_2)K_n(yR_1) - I_n(yR_1)K_n(yR_2) \right] \\
\times \left[ I_n(yR_2)K_n'(yR_1) - I_n'(yR_1)K_n'(yR_2) \right].
\]
(24)

We continue by considering \( \xi \ll 1 \) corresponding to the case when the dielectric (magnetic) properties of the surrounding medium and the layer do not differ much. Then, a series expansion of Eq. (21) with respect to \( \xi \) can be constructed. We retain only the lowest terms in the expansion \(-\xi^2\). After taking into account that \( I_n = I_{-n} \) and \( K_n = K_{-n} \), the energy \( E_c \) is rewritten in the following way:
\[
E_c \approx \frac{\hbar c^2}{4\pi} \lim_{s \to -1} [Y_1(s) + Y_2(s)],
\]
(25)

where the appropriate terms have been regrouped. \( Y_1(s) \) collects the \( n=0 \) terms from the summation
\[
Y_1(s) = \int_0^{\infty} dy y^{s-3} \left[ 4y^2 R_1 R_2 I_0(yR_1) I_0''(yR_1) K_0(yR_2) K_0''(yR_2) \\
- 2 \left( \frac{1}{R_1^2} + \frac{1}{R_2^2} \right) (y^2 I_0(y) K_0'(y) K_0'(y) + 1) \\
+ e^{2y(R_1-R_2)} \right].
\]
(26)

\( Y_2(s) \) collects all \( n \neq 0 \) terms from the summation
\[
Y_2(s) = \sum_{n=1}^{\infty} \int_0^{\infty} dy y^{s-3} \left[ 8y^2 R_1 R_2 I_n(yR_1) I_n''(yR_1) K_n(yR_2) K_n''(yR_2) \\
- 4 \left( \frac{1}{R_1^2} + \frac{1}{R_2^2} \right) y^2 I_n(y) K_n'(y) K_n'(y) \\
+ 2e^{2y(R_1-R_2)} \right].
\]
(27)

We start by evaluating the second term in \( Y_2(s) \). We make the change of variables \( y = n \beta, n = 1, 2, 3, \ldots \) and perform the uniform asymptotic expansion valid for the Bessel functions of large order [60]. Retaining the first three terms \((-1/n^2)\) leads to the expression
Here, the sum $2\sum_{n=1}^{\infty} n^{-1-s}$ is nothing but the Riemann \( \zeta \)-function, \( \zeta(1+s) \). We have added 1 to this sum, and to compensate for this, the appropriate terms are subtracted later from the \( Y_1(s) \) expression (\( n=0 \) terms). In this way, one is able to perform the regularization procedure of taking the limit \( s \to -1 \) as follows

\[
\left( \sum_{n=1}^{\infty} n^{-1-s} + 1 \right) \Gamma\left( \frac{s+1}{2} \right) = \lim_{s \to -1} \left[ (2\zeta(s+1) + 1)\Gamma\left( \frac{s+1}{2} \right) \right] = \lim_{s \to -1} \left[ 2\zeta(0) + 2\zeta'(0)(s+1) + O((s+1)^2) + 1 \right] \times \left[ \frac{2}{s+1} - \gamma + O(s+1) \right] = -2\ln(2\pi),
\]

where \( \gamma \) is the Euler constant and \( O(s+1) \) stands for the higher-order terms. In the above expression, \( \zeta(1+s) \) is expanded in Taylor series and \( \zeta(0) = -\frac{1}{2}, \zeta'(0) = -\frac{1}{2}\ln(2\pi) \) are taken into account. A similar procedure was applied in Ref. [5] to remove the divergences from the poles of the \( \Gamma \) function in calculating the Casimir energy for a dielectric cylinder.

Next, we calculate the first term in Eq. (27) by making the same change of variables \( y = n^a, n = 1, 2, 3, \ldots \) After retaining the dominant term in the uniform asymptotic expansion of each Bessel function, we obtain

\[
8\sum_{n=1}^{\infty} \int_0^\infty y^{-s} R_1 R_2 I_n(y R_1) I_n'(y R_2) K_n(y R_2) K_n'(y R_2) dy
\]

\[\equiv -2\sum_{n=1}^{\infty} n^{-1-s} \int_0^\infty y^{-s} e^{2n(\eta_1 - \eta_2)} dy.\]

where \( \eta_i = \sqrt{1 + (n R_i)^2} + \ln(R_i y / [1 + \sqrt{1 + (n R_i)^2}]) \) for \( i = 1, 2 \). Interchanging the integration and the summation in the above expression and taking the limit \( s \to -1 \), one obtains

\[
\int_0^\infty dy \frac{y}{4\sqrt{\eta_1 - \eta_2}} \sum_{n=1}^{\infty} e^{2n(\eta_1 - \eta_2)}
\]

\[= \int_0^\infty dy \frac{y e^{2n(\eta_1 - \eta_2)}}{(e^{2n(\eta_1 - \eta_2)} - 1)}.
\]

The first term in Eq. (27) can be solved exactly in the \( s \to -1 \) limit, giving the result

\[
2\sum_{n=1}^{\infty} \int_0^\infty dy R_1 y I_n(y R_1) I_n'(y R_2) K_n(y R_2) K_n'(y R_2)
\]

\[\equiv \frac{\zeta(0)}{2(R_1 - R_2)^2}.
\]

Next, we consider \( Y_1(s) \) from Eq. (26). Only the last term can be evaluated analytically, and the rest of the terms can be calculated numerically. After the limit \( s \to -1 \) is taken, one obtains

\[
Y_1 = \int_0^\infty dy \left\{ 4y^3 R_1 R_2 I_n(y R_1) I_n'(y R_2) K_n(y R_2) K_n'(y R_2)
\]

\[\equiv \frac{1}{R_1} + \frac{1}{R_2} \left\{ \frac{2y^2 I_o(y R_1) K_o(y R_2) K_o'(y R_2)}{\left[ 2y^2 + \frac{2}{(1 + y^2)^2} + \frac{1}{(1 + y^2)^3} \right] + \frac{y}{2} \right\},
\]

\[\equiv \frac{1}{4(R_1 - R_2)^2}.
\]

All terms in Eq. (29) and Eqs. (31)–(33) are convergent. Finite and physically meaningful values for \( E_C \) for different radii \( R_1, R_2 \) of the layer can be obtained from the expression

\[
E_C \equiv \frac{\hbar c^2}{2\pi} \int_0^\infty dy \frac{y e^{2n(\eta_1 - \eta_2)}}{(e^{2n(\eta_1 - \eta_2)} - 1)^3}
\]

\[+ 2y^3 R_1 R_2 I_o(y R_1) I_o'(y R_2) K_o(y R_2) K_o'(y R_2)
\]

\[\equiv \frac{1}{R_1} + \frac{1}{R_2} \left\{ \frac{2y^2 I_o(y R_1) K_o(y R_2) K_o'(y R_2)}{\left[ 2y^2 + \frac{2}{(1 + y^2)^2} + \frac{1}{(1 + y^2)^3} \right] + \frac{y}{2} \right\},
\]

\[\equiv \frac{2}{8} \left( \frac{1}{R_1} + \frac{1}{R_2} \right).
\]

Thus we were able to successfully apply the mode summa-
tion method for calculating the Casimir self-energy of a dielectric-diamagnetic cylindrical layer of finite thickness by removing the unphysical divergences with the Riemann $\zeta$-function regularization procedure.

Before the numerical evaluation of Eq. (34) is given, we investigate $E_C$ for various limits of the layer dimensions. In the limit of a layer with small thickness $\alpha \rightarrow 1$, where $\alpha = R_2/R_1$, the dominant contribution to the Casimir energy comes from Eq. (31). One finds that in this case the Casimir energy per unit length is $E_C = -0.0766 c \xi^2 R_1^2 (\alpha - 1)^3$. In the limit of $R_1, R_2 \rightarrow \infty$ with $d = R_2 - R_1 = \text{const}$, one finds that $E_C/2 \pi R_1 \approx -h c \xi^2/(8 \pi^2 d^3)$, thus precisely reproducing the Casimir energy per unit area of an infinite dielectric-diamagnetic plate [61].

Also, for the case of large $\alpha$ which corresponds to $R_2 \rightarrow \infty$ only the terms $-1/R_2^2$ in the expressions for $Y_1$ and $Y_2$ contribute to $E_C$. The problem actually becomes equivalent to a compact dielectric cylinder with radius $R_1$ and dielectric and magnetic properties $(\varepsilon, \mu)$ imbedded in an infinite medium with dielectric and magnetic properties $(\varepsilon_m, \mu_m)$ with the speed of light being constant across the interface at $R_1$. For such a dielectric-diamagnetic compact cylinder, it was found that $E_C = 0$ for the first term ($-\xi^2$) in the $\xi$ series [3,5]. Here, the terms $-1/R_2^2$ from Eqs. (27) and (33) give a finite positive contribution. However, we evaluated the next term ($-1/n^6$) in the Bessel functions asymptotic uniform expansion in the calculation of Eq. (28). We found that this term is essential, because due to the occurring cancellations the value of the Casimir energy in the limit of large $\alpha$ becomes zero, thus recovering the results reported in Refs. [3,5]. The calculations are very similar to the ones described so far, and they are not given here.

C. Numerical results

In this section we present some numerical results for the Casimir energy based on the considerations in the previous section. In Fig. 2(a), $E_C$ is shown as a function of $R_2/R_1$.

![Figure 2](image-url)

**FIG. 2.** (Color online) (a) The dimensionless Casimir energy per unit length for a cylindrical dielectric-diamagnetic layer as a function of $R_2/R_1$. (b) The dimensionless Casimir energy per unit length for the same layer as a function of inner radius $R_1$.

The figure indicates that the energy is negative, and $|E_C| \sim 1/(\alpha - 1)^3$ for $\alpha \rightarrow 1$. Also for $\alpha \approx 7$ the layer is already in the limit of large $\alpha$ and $E_C \rightarrow 0$, thus the system behaves as a compact cylinder with $(\varepsilon, \mu)$ characteristics imbedded in an infinite medium with $(\varepsilon_m, \mu_m)$ characteristics.

In Fig. 2(b), we show the Casimir energy as a function of the size of the layer for several choices of its thickness. It is seen that the most dramatic changes are found for smaller thicknesses attributed to the dominant contribution from Eq. (31). As the thickness is increased, $R_2/R_1$ increases, and the value of $E_C$ approaches that of a compact cylinder with radius $R_1$. Thus, in this case much smaller values for $|E_C|$ and much less dramatic changes in the functional dependence of the Casimir energy in terms of $R_1$ are found.

D. $\xi = (\varepsilon - \varepsilon_m)/(\varepsilon + \varepsilon_m) = 1$ case

Here we evaluate the Casimir energy for the case of $\xi = 1$. This corresponds to the situation where $\varepsilon \gg \varepsilon_m$. If in addition one takes $c = 1$, then the $\xi = 1$ case describes a perfectly conducting cylindrical shell (see Ref. [3]). The calculation proceeds from Eq. (21) by first subtracting the Casimir energies for two single cylinders with radii $R_1$ and $R_2$. This is done in order to express the answer for $E_C(s)$ later on in a more convenient and transparent way.

$$
\bar{E}_C(s) = E_C(s) - E_C^{(1)}(s) - E_C^{(2)}(s), \quad (35)
$$

where $E_C(s)$ is given in Eq. (21), and $E_C^{(1,2)}(s)$ are the Casimir energies for a single cylinder with radius $R_1, R_2$, respectively. Note that $E_C^{(1,2)}(s)$ has already been derived in Refs. [3,5], obtaining the final form $\bar{E}_C(s \rightarrow -1) = -0.01356 c \xi R_1^2/2$.

Next, Eq. (35) can be rewritten in the form

$$
\bar{E}_C(s) = \frac{\hbar c \xi s}{4 \sqrt{\pi} \Gamma \left( \frac{s}{2} \right) \Gamma \left( \frac{3}{2} - \frac{s}{2} \right)} \sum_{n=0}^{\infty} \int_{0}^{\infty} dy y^{1-s} \frac{dy}{dy} \ln \left( \frac{f_n^{TE}(iR_1, iR_2)}{f_n^{TM}(iR_1, iR_2)} \right) f_n^{TM}(iR_1)(\xi) f_n^{TM}(iR_2)(\xi))^2, \quad (36)
$$
where \( f_{n}^{TE, TM}(iR_{1}y, iR_{2}y) \) are the dispersion relations defined in Eq. (22) with \( \xi=1 \). In this case only the term containing the \( X_{2} \) expression [Eq. (24)] is important, the rest are zero. \( X_{2} \) actually is the dispersion relation for the electromagnetic modes for two infinitely thin concentric perfectly conducting cylindrical shells [4, 49]. The TE modes are determined by the parts containing the \( I_{n}(x) \) functions, while the TM modes are determined by the parts containing the \( K_{n}(x) \) functions. \( f_{n}^{TE, TM}(iR_{1}y) \), \( f_{n}^{TE, TM}(iR_{2}y) \) are the dispersion relations for a single solid cylinder with radii \( R_{1}, R_{2}, \) respectively. They were derived in Refs. [3, 5], where it was shown that when \( \xi=1 \) and \( c=1 \), \( f_{n}^{TE, TM}(iR_{1}y) \) become equivalent to the dispersion relations of a single infinitely thin perfectly conducting shell. \( f_{n}^{TM}(i\infty) \) are the dispersion relations with no boundaries for a single cylinder.

Substituting the expressions for the dispersion relations in the case of \( \xi=1 \) in the limit of \( s\rightarrow-1 \), one obtains

\[
E_{C} = \frac{\hbar c}{4\pi} \left[ \int_{0}^{\infty} y \ln[(1 - \bar{I}_{0}K_{0})(1 - \bar{I}_{0}K_{0}')] dy + \sum_{n=1}^{\infty} \int_{0}^{\infty} y \ln[(1 - \bar{I}_{n}K_{n})(1 - \bar{I}_{n}K_{n}')] dy \right],
\]

where \( \bar{I}_{n} = I_{n}(y_{1})/I_{n}(y_{2}), \bar{I}_{n}' = I_{n}'(y_{1})/I_{n}'(y_{2}), \bar{K}_{n} = K_{n}(y_{1})/K_{n}(y_{2}), \) and \( \bar{K}_{n}' = K_{n}'(y_{1})/K_{n}'(y_{2}) \) for \( n=0, 1, 2, \ldots \) with \( y_{1}=y_{R_{1}} \) and \( y_{2}=y_{R_{2}} \). A similar formula has already been derived in Ref. [4] where the Casimir energy for two conducting cylindrical shells was calculated using the exponential cutoff regularization technique. The first term in Eq. (37) is evaluated numerically. The second term is calculated using the uniform asymptotic expansion of the Bessel functions for large orders and the Taylor series of the logarithmic function \( \ln(1-x) \) for \( x<1 \), where \( x=e^{-2|y_{1}-y_{2}|} \),

\[
2 \sum_{n=1}^{\infty} \int_{0}^{\infty} y \ln[(1 - \bar{I}_{n}K_{n})(1 - \bar{I}_{n}K_{n}')] dy \approx -4 \sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{y}{m(e^{2|y_{1}-y_{2}|} - 1)} dy.
\]

The expression for the Casimir energy becomes

\[
E_{C} = \frac{\hbar c}{4\pi} \left[ \int_{0}^{\infty} y \ln[(1 - \bar{I}_{0}K_{0})(1 - \bar{I}_{0}K_{0}')] dy - 4 \sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{y}{m(e^{2|y_{1}-y_{2}|} - 1)} dy \right] - 0.01356 \hbar c \left( \frac{1}{R_{1}^{2}} + \frac{1}{R_{2}^{2}} \right).
\]

Each integral from the sum in Eq. (39) can be evaluated numerically, and the series over \( m \) is found to rapidly converge [4]. In Fig. 3 we show the Casimir energy for the conducting shells as a function of \( a=R_{2}/R_{1} \). In the limit of \( a\rightarrow1 \) corresponding to the two cylinders being close to each other, Fig. 3 can be compared to the analytical expression \( E_{C}\approx -0.0862\hbar c/[R_{1}^{2}(\alpha-1)^{3}] \) which we obtain from Eq. (39).

In the limit of \( R_{1}, R_{2}\rightarrow\infty \) with \( d=R_{2}-R_{1}\) const, Eq. (39) results in the well known formula \( E_{C}/2\pi R_{1}^{2} = -\hbar c \pi^{2}/(720d^{5}) \) for the Casimir energy per unit area of two perfectly conducting plates [29].

Furthermore, Fig. 3 shows that the energy becomes constant in the limit of large \( a \). In this case only the term \( -1/R_{1}^{2} \) in Eq. (39) is nonzero, giving the result for one perfectly conducting infinitely thin cylindrical shell. Figure 3 shows that in the intermediate region of \( 1<\alpha<3 \), the Casimir energy for the dielectric-diamagnetic layer is smaller (less negative) as compared to the one for the two perfectly conducting shells.

**IV. CONCLUSIONS**

In this paper we have calculated the Casimir interaction energy for a cylindrical layer with a finite thickness using the zero-point energy summation technique and the Riemann \( \zeta \)-function regularization procedure. The dielectric and magnetic properties of the layer and those of the surrounding medium are assumed to be described by real constant dielectric permittivities and magnetic permeabilities satisfying the relationship of constant speed of light across the interfaces. The method proves to be convenient and intuitively easy to follow. It is formulated in terms of only one parameter \( \xi = (e-e_{m})/(e+e_{m}) \) which allows us to make analytical and numerical evaluations for practically important limiting cases such as a dielectric-diamagnetic cylindrical layer as well as two concentric perfectly conducting thin shells.

We considered the case of a dielectric-diamagnetic layer with \( \xi=1 \) and evaluated the first nonzero term \( \propto\xi^{2} \) in the infinite sum representing the Casimir energy per unit length. The Casimir energy is negative, and it is \( \propto-1/(R_{2}/R_{1}-1)^{3} \) in the limit \( R_{2}/R_{1}\rightarrow1 \). When \( R_{2}\rightarrow\infty \), the problem becomes equivalent to that of a dielectric-diamagnetic solid cylinder. The Casimir energy \( \propto\xi^{2} \) for such a cylinder is zero [3, 5], and here we recover this result for \( R_{2}\rightarrow\infty \). We also recover the Casimir energy per unit area of a dielectric-diamagnetic plate [61] in the limit of \( R_{1}, R_{2}\rightarrow\infty \) when \( d=R_{2}-R_{1}\) const.

We also analyzed the case of \( \xi=1 \) describing two thin perfectly conducting concentric cylindrical shells. We find that the Casimir energy in this case is also negative and it is \( \propto-1/(R_{2}/R_{1}-1)^{3} \) for \( R_{2}/R_{1}\rightarrow1 \), whereas for \( R_{2}\rightarrow\infty \) it approaches the limit of a single perfectly conducting cylindri-
cal shell [3,5]. We also recover the well-known Casimir formula for the energy per unit area of two parallel perfectly conducting plates [29] separated by a distance \( d = R_2 - R_1 = \text{const} \) in the limit of \( R_1, R_2 \rightarrow \infty \).

The case of two perfectly conducting concentric cylinders might be of particular interest as a qualitative model of the Casimir interactions in a double-wall metallic carbon nanotube system. Previous theoretical studies, however, have shown that the perfectly conducting metallic cylinder model does not describe correctly the electrodynamic processes closely related to the Casimir interaction, such as atomic spontaneous decay [62,63], atom-nanotube van der Waal coupling [36], atomic light absorption [64], and atomic entanglement [65] near carbon nanotubes. In fact, one needs to take into account realistic electromagnetic properties and the strong modification of the photonic density of states due to the increasing role of the interface photonic modes near the nanotube surface. On the other hand, the perfect-conductor case discussed here might serve as a qualitative model of a double-wall metallic nanotube system in the limit of largely different radii, \( R_2 \gg R_1 \) (\( \alpha \gg 1 \) region in Fig. 3), as the role of the interface photonic modes decreases with increasing intertube separation. More thorough and realistic analysis is necessary to describe the Casimir interaction in double-wall (and multiwall) carbon nanotubes, which is the subject of our future publication.

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