Electromagnetic interactions between parallel dielectric-diamagnetic cylinders

K. Tatur* and L. M. Woods†
Department of Physics, University of South Florida, Tampa, Florida 33620, USA
(Received 20 August 2009; published 13 November 2009)

The interaction energy due to electromagnetic field fluctuations between two infinitely long straight parallel dielectric-diamagnetic cylinders immersed in a medium is considered. We make use of the mode summation method for the calculations. We investigate the energy dependence on the cylindrical radial curvature and dielectric response of the involved materials. It is shown that the sign of the interaction energy can be changed by a suitable choice of the dielectric properties of the involved objects. The condition for the relation of the material’s dielectric properties for repulsive interaction is obtained to be the same as the one for planar materials.

DOI: 10.1103/PhysRevA.80.050101

Electromagnetic field fluctuations between objects give rise to the Casimir force. The Casimir force is quantum mechanical in its nature, and it can couple electrically neutral materials. It has been successfully used in different geometries to calculate the Casimir energy [1–3]. Here we apply this approach for the system of two parallel cylinders. After the electromagnetic modes are determined, the zero-point energy is expressed as a contour integral in the complex frequency plane, where two infinite sums appear in the expression [2]. One of the sums is over the mode dispersion relation at a fixed value of the angular momentum, while the other is over the angular momentum itself. To remove the occurring divergences, the zero-point energy of the infinite homogeneous space is subtracted off from the energy of the system [2].

The particular system of interest consists of two infinitely long straight parallel circular cylinders of radii \( R_1 \) and \( R_2 \), with center-to-center separation \( R \). The permittivity and permeability of one cylinder are \( \varepsilon_i \) and \( \mu_i \), respectively, and for the other they are \( \varepsilon_j \) and \( \mu_j \). The cylinders are placed in an infinite medium with permittivity \( \varepsilon_0 \) and permeability \( \mu_0 \) (Fig. 1). All dielectric and magnetic characteristics are taken to be constants.

The electromagnetic modes are found by solving the Maxwell’s equations [11] with the appropriate boundary conditions across the interfaces of each cylinder. We find that for general \( \varepsilon_j, \mu_j \) the dispersion relation is difficult to evaluate. Thus we impose the condition of a constant speed of light \( c \) across each interface, \( c = \varepsilon_0 \mu_0^{-1/2} \) with \( j = 1, 2, 3 \) in order to facilitate the calculation of the Casimir energy. This corresponds to a dielectric-diamagnetic system meaning that there is no preferential response to the electric or magnetic field [2].

There are two cylindrical coordinate systems associated with each cylinder—\((\rho, \theta, z)\) with \( i = 1, 2 \). The electric and magnetic fields along the \( z \) axis for each region are expressed as

\[
E^{(i)}_z = \sum_{n=-\infty}^{\infty} \left\{ A^{(i)}_n J_n(\chi_0 \rho) e^{in\theta} + B^{(i+2)}_n H^{(1)}_n(\chi_0 R) \times J_n(\chi_0 \rho) e^{in\theta} \right\} f_{k_{z,ra}}, \quad (1)
\]

\[
B^{(i)}_z = \sum_{n=-\infty}^{\infty} \left\{ C^{(i)}_n J_n(\chi_0 \rho) e^{in\theta} + D^{(i+2)}_n H^{(1)}_n(\chi_0 R) \times J_n(\chi_0 \rho) e^{in\theta} \right\} f_{k_{z,ra}}, \quad (2)
\]
E_{z}^{(3)} = \sum_{n=\infty}^{\infty} \{B_{n}^{(4)} H_{n}^{(4)}(\chi_{R})e^{in\theta_{R}} + D_{n}^{(3)} H_{n}^{(1)}(\chi_{R})e^{in\theta_{R}}\}f_{z,w}, (3)

B_{z}^{(3)} = \sum_{n=\infty}^{\infty} \{D_{n}^{(4)} H_{n}^{(1)}(\chi_{R})e^{in\theta_{R}} + D_{n}^{(3)} H_{n}^{(1)}(\chi_{R})e^{in\theta_{R}}\}f_{z,w}, (4)

where \(k_{z}\) is the wave vector along the \(z\) direction, \(\omega\) is the frequency of the electromagnetic excitations, \(f_{x,w} = e^{i(k_{z}z-wt)}\), and \(\chi_{R}^{2} = \varepsilon_{\mu} \omega^{2} - k_{z}^{2}\). \(J_{n}(\chi_{R})\) and \(H_{n}^{(1)}(\chi_{R})\) are the Bessel functions of first kind of order \(n\) and the Hankel functions of the first kind of order \(n\), respectively. The \(z\) components of the electric \(E_{z}^{(1)}\) and magnetic \(B_{z}^{(3)}\) fields are expressed in the specific coordinate set of each cylinder. More specifically, we have made use of the addition theorem \([12]\) relating Bessel functions in one coordinate system with respect to another in the second terms of Eqs. (1) and (2). The electric \(E_{z}^{(3)}\) and magnetic \(B_{z}^{(3)}\) fields for the medium contain one term expressed in the coordinate set of one cylinder and another term—in the coordinate system of the other cylinder. The \(\rho\) and \(\theta\) components of the fields can be easily obtained using Maxwell’s equations \([11]\).

The unknown coefficients \(A_{n}^{(1)}, B_{n}^{(1)}, C_{n}^{(1)}, D_{n}^{(1)}, B_{n}^{(2)}, D_{n}^{(2)}\) are related by imposing the boundary conditions for the continuity of \(E_{\rho}^{(1)}, E_{\theta}^{(1)}, E_{z}^{(1)}, B_{z}^{(1)} / \mu_{z}\) across the interface of each cylinder giving the dispersion relation for the electromagnetic modes supported by this system. In the general case, the dispersion relation is complicated and the calculations of the interaction energy are not feasible, however significant simplifications occur when the speed of light \(c\) is constant everywhere. After applying the boundary conditions and keeping \(\varepsilon_{\mu} \omega^{2} = c^{2}\), the electromagnetic modes for the interaction are obtained by the dispersion relations \(f_{1}(\nu, R_{1}, R_{2}, R) = \text{Det}(1 - \Gamma) = 0\) and \(f_{2}(\nu, R_{1}, R_{2}, R) = \text{Det}(1 - \tilde{\Delta}) = 0\) where the substitution \(\nu = \text{Im} \chi\) has been made. \(\Gamma\) and \(\tilde{\Delta}\) are matrices with elements:

\[
\Gamma_{ij} = \sum_{j=\infty}^{\infty} \left[ \begin{array}{cc}
\frac{(e_{3} - e_{1})(e_{3} - e_{2})}{I_{n}(v_{R})} & K_{n}(v_{R})
\end{array} \right],
\]

\[
\Delta_{ij} = \sum_{j=\infty}^{\infty} \left[ \begin{array}{cc}
\frac{(e_{3} - e_{1})(e_{3} - e_{2})}{I_{n}(v_{R})} & K_{n}(v_{R})
\end{array} \right],
\]

where \(I_{n}(x)\) and \(K_{n}(x)\) are the modified Bessel functions of order \(n\), \(j\), respectively, and \(I_{n}^{'}(x) = dI_{n}(x)/dx\) and \(K_{n}^{'}(x) = dK_{n}(x)/dx\).

The zero-point energy is calculated using \(E_{c} = \frac{\hbar}{2\pi} \sum_{\{\rho\}} \langle \omega_{\rho} - \tilde{\omega}_{\rho} \rangle [2,3,13]\). \(\omega_{\rho}\) are the eigenfrequencies satisfying the dispersion relations and \(\tilde{\omega}_{\rho}\) are the ones corresponding to the reference vacuum with no boundaries present, \(R_{1,2} \to \infty\) (Fig. 1). Here \(\{\rho\}\) are the complete set of quantum numbers determined by the type of system under consideration. In the case of a cylindrical structure, \(\{\rho\} = (n,m,k_{z})\), where \(n\) is the order of the Bessel functions, \(m\) denotes the number of roots of the dispersion relations, and \(k_{z}\) is the continuous wave along the \(z\) axis. The interaction energy per unit length of the cylindrical system can then be expressed as \(E_{C} = \frac{\hbar}{8\pi} \int_{0}^{\infty} d\nu \sqrt{\frac{d}{d\nu}} \left\{ \sum_{n=\infty}^{\infty} \left[ \sum_{j=\infty}^{\infty} (\Gamma_{n} + \Delta_{n} - \Gamma_{n}(\nu) - \Delta_{n}(\nu)) \right] \right\}\).

We examined the various terms occurring in Eq. (8), and find it convenient to arrange them into \((n=0,j=0), (n=0,j \neq 0), (n \neq 0,j=0), (n \neq 0,j \neq 0)\) groups. The terms with \(n \neq 0, j \neq 0\) are evaluated using large order expansion for the Bessel functions \([12]\). This is a technique used in other
works for the interaction energy in cylindrical structures [2,3,13]. We obtain the following convergent expression for the zero-point energy per unit length \( l \), for the two parallel cylinders:

\[
E_C = -\frac{\hbar c}{4\pi} \left[ \int_0^\infty dv v^2 \left( \frac{(e_3-e_1)(e_3-e_2)K_0^2(vR)}{e_2I_0^2(vR_2) - e_3I_0^2(vR_2)} \right) + \frac{2}{\pi} \frac{(e_3-e_1)(e_3-e_2)}{(e_3+e_1)(e_3+e_2)} \int_0^\infty dv \frac{\alpha^2 - \beta^2}{v^2} \right],
\]

where \( \eta = 1 + (1/R)^2 + \ln[R/v + 1 + (1/R)^2] \) and \( \eta_{1,2} = 1 + (R_{1,2}/v)^2 + \ln[R_{1,2}/v + 1 + (1/R_{1,2})^2] \). Further, we consider the situation of two largely separated cylinders (\( R \gg R_1, R_2 \)). This particular limit can be used as a qualitative model to investigate the interaction of two tubular systems separated by large distances. To simplify the evaluations, we take the cylinders to have equal radii \( R_1 = R_2 \) and the same dielectric \( e_1 = e_2 \) and magnetic \( \mu_1 = \mu_2 \) properties. When \( R \gg R_1 = R_2 \), the first two terms in Eq. (9) are approximated by taking a small expansion with respect to \( v/R_{1,2} \) in the denominators. The integrals in these terms then become of the form \( \int_0^\infty dv v^p K_0^2(vR) \) with \( p = 5,7 \) and \( \int_0^\infty dv v^{p-1} K_0^2(vR) / q = 8 \), which can be easily evaluated [15]. The rest of the terms in Eq. (9) can also be simplified by realizing that for large \( R \), \( \eta - \eta_1 \approx vR_1 + \ln[2/vR_1] \). The integrals in the third and fourth terms become \( \sim \int_0^\infty dv (v/R_{1,2})^2 e^{-2vR} \), and the integrals in the last term in Eq. (9) become \( \sim \int_0^\infty dv (v/R_{1,2})^{2(p+1)} e^{-2vR} \), which can also be evaluated [15]. We find that the series over \( n \) and \( j \) rapidly converges, and only the first few terms are dominant. Thus in the limit of \( R \gg R_1 = R_2 \), we find that the energy per unit length is expressed as a sum of terms proportional to powers of the \( R_1/R \) ratio. The first two dominant terms are obtained to be

\[
E_C = -\frac{\hbar c}{8\pi} \frac{1}{(e_3-e_1)_2} \left[ \frac{1}{2} \frac{R_1^2}{e_1 + e_2} \right] + \frac{1}{512\pi} \frac{1}{(e_3+e_1)^2} \left[ \frac{R_1^2}{R_{1,2}^2} \right].
\]

Other terms in \( E_C \) have higher powers of \( R_1/R_2^2 \) indicating that their contribution is small.

To investigate further the dependence of the interaction energy on the radial dimensions of the cylinders and their separation, we evaluate Eq. (9) numerically. Figure 2 shows the results for the interaction energy as a function of surface-to-surface separation \( d \) with \( R_1 = R_2 = 1 \) nm. One sees that as \( R_2 \) is increased the energy diverges slower as a function of \( d \) when compared to the one for \( R_1 \sim R_2 \). At larger \( d \) separations, \( E_C \) decreases slower when \( R_2 \gg R_1 \) as displayed in the inset of Fig. 2. The crossover occurs at \( d \sim 0.4 \) nm. The calculations reveal that the dominant contribution for the cases of \( R_1 \sim R_2 \) comes from the third and fourth terms of Eq. (9). As \( R_2 \) becomes larger, the fourth term in Eq. (9) remains dominant, while the third one becomes less important. The reason for the crossover (inset in Fig. 2) for small and large \( R_2 \) can be traced to the exponential distance dependences in these terms. There is a faster decrease in \( E_C \) for small \( d \) when \( R_2 \gg R_1 \) as compared to the cases of small \( d \) and \( R_1 \sim R_2 \) due to the faster decrease in the exponential factors. But for large \( d \), the situation is reversed as indicated by the inset of Fig. 2. When \( d \sim R_2 \gg R_1 \) there are two large distance scales. Because of their dominance through the \( \eta \) and \( \eta_{1,2} \) factors (which are subtracted), the exponential decrease in the fourth term is actually slower than the case for \( d \gg R_2 \sim R_1 \). The results shown in Fig. 2 are independent of the choices for the dielectric responses of the cylinders and medium.

In addition to the radial size of the cylinders and the distance between them, the dielectric response properties of the cylinders and the environment are another important factor in the interaction energy. It has been predicted that the interaction between planar materials 1 and 2 immersed in medium 3 can be repulsive if the value of the dielectric constant of the medium is between the values of the dielectric constants of the materials [16]. Recent measurements of the Casimir force between a large sphere and a plate covered...
with a layer of silica, for which this condition for the dielectric properties is satisfied, demonstrate that the interaction can be repulsive [17].

Our calculations show that repulsive interaction can exist between cylindrical structures also. The obtained results provide further evidence that the sign and the magnitude of the Casimir force can be manipulated simply by choosing materials with certain dielectric response properties for objects in the cylindrical geometry as well. Figure 3 shows how \( E_C \) becomes negative, thus \( E_{C} \) is attractive. This is what is shown in Fig. 3(b) shows that for \( \varepsilon_2=2 \) and \( \varepsilon_1=5 \) \( E_C >0 \) as long as \( \varepsilon_1 > \varepsilon_2 > \varepsilon_3 \) is satisfied. Also, for \( \varepsilon_2=7 \) and \( \varepsilon_1=3 \), \( E_C >0 \) when \( \varepsilon_2 > \varepsilon_1 > \varepsilon_3 \) is satisfied. Furthermore, the prefactor \( \xi \) controls the magnitude of the interaction energy. When the dielectric constants of the cylinders and the environment are such that \( \varepsilon_2 - \varepsilon_3 \) and \( \varepsilon_2 - \varepsilon_1 \) are small, \( |E_C| \) is also small in magnitude. However, when the dielectric properties are very different in value, then \( |E_C| \) becomes large.

In conclusion, we have demonstrated that the Casimir energy for a system of two infinitely long parallel cylinders immersed in medium can be calculated using the mode summation method. To our knowledge, this is the first application of this method to parallel cylinders since most previous works deal with only one cylinder or concentric cylinders [2–4]. The mode summation method proves to be convenient for the derivation of finite analytical expressions for the interaction energy when a dielectric-diamagnetic system is considered. Our results for the energy dependence on the radial size and curvature can be potentially of significant importance as a qualitative model of the interaction due to electromagnetic excitations between tubular materials, such as carbon nanotubes and nanowires.

The analytical expressions we have derived are of particular importance for the Casimir energy dependence on the dielectric properties of the involved objects. We show that the interaction energy is positive (repulsive force) when the value of the dielectric constant of the medium is between the values of the dielectric constants of the two cylinders. This particular result may be of significance to research efforts in quantum levitation or reducing friction between nanosized components in devices when cylindrical objects are involved. It is interesting to note that the relation \( \varepsilon_1 > \varepsilon_3 > \varepsilon_2 \) is the same for a repulsive force between planar systems as described in [17]. Thus our calculations may be viewed as further evidence that the sign of the Casimir force can be manipulated by changing the dielectric response properties of the involved objects regardless of their geometry.

We acknowledge financial support from the Department of Energy under Contract No. DE-FG02-06ER46297.