Nonlinear electron-phonon heat exchange

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A calculation of the energy exchange between phonons and electrons is done for a metal at very low temperatures. We consider the energy exchange due to two-phonon processes. Second-order processes are expected to be important at temperatures less than 1 K. We include two different second-order processes: (i) the Compton-like scattering of phonons, and (ii) the electron-dual-phonon scattering from the second-order electron-phonon interaction. It is found that the Compton-like process contains a singular energy denominator. The singularity is removed by introducing quasiparticle damping. For pure metals we find that the energy exchange depends upon the lifetime of the electrons and it is proportional to the temperature of the lattice as $T_L^2$. The same calculation is performed for the electron-dual-phonon scattering and it is found that the temperature dependence is $T_L^4$. The results can be applied to quantum dot refrigerators. [S0163-1829(98)02114-6]

I. INTRODUCTION

The electron-phonon interaction is an important mechanism for the heat exchange between electrons and phonons. Much attention has been paid to the contribution from the first-order scattering processes, when the electron emits or absorbs a single phonon. It is found by Allen\(^1\) that in a metal the thermal rate is proportional to $(T_L^2 - T_e^5)$ where $T_L$ is the lattice temperature and $T_e$ is the electron temperature.

There have been several recent proposals\(^2\) and actual constructions\(^4\) of low-temperature quantum dot refrigerators in metals and semiconductors. These are all based upon the idea of boiling off the energetic electrons, leaving behind the “cooler” electrons. In this way one can cool the electrons to a lower temperature. The amount of cooling depends on the rate of heat exchange between the electrons and the thermal bath, which are phonons. One-phonon heat exchange predicts that heat exchange is proportional to $(T_L^2 - T_e^5)$. Here we consider the rate of heat exchange of two-phonon processes. We assume a metallic environment at low temperature, and calculate how much energy is exchanged between the electrons and the phonons. The motivation is provided by transport theory that shows that two-phonon processes become important at low temperature, in both semiconductors\(^5\) and metals.\(^6\) We include both the Comptons of phonon scattering, as well as the second-order process from dual-phonon scattering.\(^6\)

Some experiments have been done using a refrigeration scheme\(^4\) that show that by manipulating the Fermi-Dirac distribution the electrons can be cooled below the lattice. For temperatures higher than 1 K a simple theory using the BCS density of states can be applied. But when the temperature is lower than 1 K, the measurements show that the electron energies differ strongly from an equilibrium distribution. It is suggested that the processes that are very important at this temperature region are second-order scattering processes,\(^5\) not absorption or emission of a single phonon. The higher-order scattering processes were suggested by Enz.\(^7\)

The interaction Hamiltonian between electrons and phonons can be expanded in powers of the lattice displacement:

$$\hat{H}_{ep} = \hat{H}_{ep}^1 + \hat{H}_{ep}^2,$$

$$\hat{H}_{ep}^1 = -\sum_n \mathbf{u}_n \cdot \nabla V(\mathbf{r} - \mathbf{R}_n),$$

$$\hat{H}_{ep}^2 = \frac{1}{2} \sum_n \mathbf{u}_n \cdot \nabla \nabla V(\mathbf{r} - \mathbf{R}_n) \cdot \mathbf{u}_n,$$

where $\mathbf{r}$ is the vector position of electron, $\mathbf{R}_n$ is the position of the $n$th ion in equilibrium, and $\mathbf{u}_n$ is the displacement of the $n$th ion. The first-order term $\hat{H}_{ep}^1$ gives rise to the familiar one-phonon scattering. The second-order term $\hat{H}_{ep}^2$ and higher-order corrections correspond to multiphonon scattering in which two or more phonons are created and/or destroyed. However, two-phonon scattering can also arise by a repeated application of $\hat{H}_{ep}^1$ through intermediate states.

Interference of the matrix elements of $\hat{H}_{ep}^2$ and the second-order ones of $\hat{H}_{ep}^1$ may result in some cancellation. For the case of long-wavelength acoustic-phonon modes the results of interference and cancellations are drastic, as shown by Herring.\(^8\) In the limit of long wavelength an acoustic phonon is equivalent to translation of the lattice. The cancellation in this special case of interaction reflects the translational invariance of the crystal lattice. Therefore, one may think that there is no need to calculate the effects of second-order scattering processes.

Usually the electron-phonon heat exchange is calculated from electron scattering by a single phonon. Then one can take high-temperature limit or low-temperature limit in accordance with a particular interest.\(^1\) However, our calculations show that at very low temperatures the second-order electron-phonon processes for the heat exchange are important and the results cannot be neglected.

In this paper we consider (i) Compton-like scattering of phonons and (ii) electron-dual-phonon scattering from the second-order matrix elements of $\hat{H}_{ep}^1$. The involvement of
second-order processes makes it much harder to calculate the heat exchange. We use the Matsubara Green’s function method to find the electron self-energy for these processes. It is found that for the Compton-like scattering the thermal relaxation depends on the temperature of the lattice according to $T^6$. We also have found that there is a singular energy denominator. The singularity is removed by introducing a quasiparticle damping term. The final result depends on the lifetime of the electrons. Some measurements of electron-quasiparticle damping term. The final result depends on the parameter $\alpha = T_L/T_e$.

II. COMPTON-LIKE SCATTERING

Compton-like processes are very likely to occur at low temperatures. The scattering can be described as follows. An electron emits (absorbs) a phonon making a transition into a virtual intermediate state and then absorbs (emits) another phonon. To calculate the heat exchange mathematically we start with the interaction Hamiltonian

$$ H_{\text{int}} = \sum_q \sum_p M(q) C^+_p q C_p(a_q+a_{-q}^+), $$

with displacement $u_n$ expressed as follows:

$$ u_n = \sum_q \left( \frac{1}{2N\Omega q} \right)^{1/2} e^{iq \cdot n} n(a_q+a_{-q}^+), $$

and a matrix element

$$ M(q) = \left( \frac{1}{2N\Omega q} \right)^{1/2} V(|q|) \hat{\xi}_q \cdot \hat{q}. $$

There are two Feynman diagrams which correspond to the two types of Compton-like scattering. They are presented in Fig. 1.

The electron and phonon Matsubara Green’s functions are defined:

$$ G^0(p,i\omega) = \frac{1}{i\omega - \hat{\xi}_p}, $$

$$ D^0(q,i\omega) = -\frac{2\omega_q}{\omega^2 + \omega_q^2}. $$

Now one can easily write the self-energies of an electron for the two Feynman diagrams using Matsubara Green’s function method:

$$ \sum_{<} (p,i\omega) = \frac{1}{\beta^2} \sum_{ow} \sum_{qq'} M^2(q) M^2(q') \frac{2\omega_q}{(\omega^2 + \omega_q^2)} \frac{2\omega_{q'}}{(\omega'^2 + \omega_{q'}^2)} \frac{1}{(i\mathbf{p} + i\omega - \xi_{p+q}^+ q')}, $$

$$ \sum_{<'} (p,i\omega) = \frac{1}{\beta^2} \sum_{ow'} \sum_{qq'} M^2(q) M^2(q') \frac{2\omega_q}{(\omega^2 + \omega_q^2)} \frac{2\omega_{q'}}{(\omega'^2 + \omega_{q'}^2)} \frac{1}{(i\mathbf{p} + i\omega - \xi_{p+q}^+ q')}. $$
Further, we consider interaction with acoustic phonons. The phonon spectrum is given by the relations
\[ \omega_q = sq, \]  
\[ \omega'_q = sq', \]  
where \( \omega \) and \( \omega' \) are the phonon energies \((h=1)\), \( q \) is the magnitude of the wave vector and \( s \) is the velocity of sound. One can perform summations over \( \omega \) and \( \omega' \) and then do analytic continuation from \( i\rho \) to \( \epsilon + i\delta \). The electron dispersion is parabolic and isotropic:
\[ \epsilon = \frac{p^2}{2m}, \]
where \( m \) is the electron effective mass and \( p \) is the electron momentum. The imaginary parts of the self-energy for the two diagrams are evaluated. We retain only the terms that pertain to two-phonon processes.

\[
\text{Im} \sum_{ \Omega } (p,\epsilon) = -\pi \int \frac{d^3q}{(2\pi)^3} M^2(q) \left( \frac{\delta(\epsilon - \omega + \omega' - \xi'\epsilon)}{(\epsilon - \omega - \xi')(\epsilon - \omega' - \xi')} \right) \left[ NN' + f''(N + N' + 1) \right] + \frac{\delta(\epsilon - \omega + \omega' - \xi'\epsilon)}{(\epsilon - \omega - \xi')^2} \left[ (N + 1)N + f''(N - N') \right] \times \left[ (N + 1)(N' + 1) - f''(N + N' + 1) \right]
\]
\[
\text{Im} \sum_{ \Omega } (p,\epsilon) = -\pi \int \frac{d^3q}{(2\pi)^3} M^2(q) \left( \frac{\delta(\epsilon + \omega + \omega' - \xi''\epsilon)}{(\epsilon + \omega + \xi')^2} \right) \left[ NN' + f''(N + N' + 1) \right] + \frac{\delta(\epsilon - \omega + \omega' - \xi''\epsilon)}{(\epsilon - \omega - \xi')^2} \left[ (N + 1)N + f''(N - N') \right] \times \left[ (N + 1)(N' + 1) - f''(N + N' + 1) \right]
\]
\[
\text{Im} \sum_{ \Omega } (p,\epsilon) = -\pi \int \frac{d^3q}{(2\pi)^3} M^2(q) \left( \frac{\delta(\epsilon - \omega + \omega' - \xi''\epsilon)}{(\epsilon - \omega - \xi')(\epsilon - \omega' - \xi')} \right) \left[ NN' + f''(N + N' + 1) \right] + \frac{\delta(\epsilon - \omega + \omega' - \xi''\epsilon)}{(\epsilon - \omega - \xi')^2} \left[ (N + 1)N + f''(N - N') \right] \times \left[ (N + 1)(N' + 1) - f''(N + N' + 1) \right]
\]

where the following substitutions are made:
\[ \xi = \epsilon + q, \]
\[ \xi' = \epsilon + q', \]
\[ \xi'' = \epsilon + q + q', \]
\[ N = n_\rho(\omega) = \frac{1}{e^{\beta\omega} - 1}, \]
\[ N' = n_\rho(\omega') = \frac{1}{e^{\beta\omega'} - 1}, \]
\[ f'' = n_f(\epsilon + q + q') = \frac{1}{e^{\beta(\epsilon + q + q')} - 1}. \]

In the second expression we make the following substitutions due to symmetry:
\[ \frac{1}{(\epsilon + \omega - \xi')^2} = \frac{1}{2} \left( \frac{1}{(\epsilon + \omega - \xi')^2} + \frac{1}{(\epsilon + \omega' - \xi')^2} \right), \]
\[ \frac{1}{(\epsilon - \omega - \xi')^2} = \frac{1}{2} \left( \frac{1}{(\epsilon - \omega - \xi')^2} + \frac{1}{(\epsilon - \omega' - \xi')^2} \right), \]
\[ \frac{1}{(\epsilon + \omega - \xi')^2} = \frac{1}{2} \left( \frac{1}{(\epsilon + \omega - \xi')^2} + \frac{1}{(\epsilon - \omega' + \xi')^2} \right), \]
\[ \frac{1}{(\epsilon - \omega + \xi')^2} = \frac{1}{2} \left( \frac{1}{(\epsilon - \omega + \xi')^2} + \frac{1}{(\epsilon - \omega' + \xi')^2} \right). \]

We notice that both expressions for \( \text{Im} \Sigma_{\Omega}(p,\epsilon) \) and \( \text{Im} \Sigma_{\Omega}(p,\epsilon) \) have the same occupation numbers and they can be combined easily. Let us note that in this model we take that the distributions \( N(\omega) \) and \( n(\epsilon) \) are equal to the local equilibrium distributions and they are characterized by separate electron and lattice temperatures \( T_e \) and \( T_L \) that depend on time. So, one can write the subsequent energy redistribution:
\( \frac{dE_e}{dt} = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3q'}{(2\pi)^3} \times M^2(q)M^2(q')(\epsilon_p - \epsilon_{p+q+q'}) \cdot I, \) (16)

where the expression for \( I \) is

\[
I = \delta(\epsilon + \omega + \omega' - \xi''\gamma) \left( \frac{1}{\epsilon + \omega - \xi} + \frac{1}{\epsilon + \omega' - \xi} \right)^2
\times \left[ (N+1)(N'+1)f(1-f'') + f''(1-f)(N+1)(N'+1) \right]
+ \delta(\epsilon - \omega - \omega' - \xi''\gamma) \left( \frac{1}{\epsilon - \omega - \xi} + \frac{1}{\epsilon - \omega' - \xi} \right)^2
\times \left[ (N+1)(N'+1)f(1-f'') - f''(1-f)N(N'+1) \right]
+ \delta(\epsilon + \omega + \omega' - \xi''\gamma) \left( \frac{1}{\epsilon + \omega - \xi} + \frac{1}{\epsilon + \omega' - \xi} \right)^2
\times \left[ (N+1)(N')f(1-f'') + f''(1-f)N(N'+1) \right]
\times \frac{\delta(\epsilon_p \pm sq \pm sq' - \epsilon_{p+q+q'})}{\pm sq' \pm sq'' - \frac{pq\nu}{m}}. \] (17)

The expressions for \( f \) and \( f'' \) are, respectively,

\[
f = \frac{1}{e^{\beta \nu q + 1}}, \]
\[
f'' = \frac{1}{e^{\beta \nu q' + 1}}.
\]

Notice the following symmetries in the formula for \( I \): (i) the first two terms count the number of final phonons twice; (ii) the last two terms are identical if one switches \( q \) and \( q' \). This is the reason for appearance of \( \frac{1}{2} \) in the final result.

It is clear from the expression for \( I \) that the squared energy denominator will give a singular result when integrated. To continue further we have to modify the integral. The usual way of doing it is to insert an imaginary correction. In this way one is able to continue the calculations.

Now we perform the actual estimation by changing the integral variables as follows:

\[
\int \frac{d^3q d^3q'}{(2\pi)^6} = \frac{1}{(2\pi)^3} \times \int_0^\infty q^2 dq \int_{-1}^1 d\nu \int_0^\nu q^2 dq' \int_{-1}^1 d\nu',
\]

where

\[
\hat{q} \cdot \hat{p} = \nu,
\]
\[
\hat{q}' \cdot \hat{p} = \nu',
\]
\[
\hat{q} \cdot \hat{q}' = \nu \nu' + \sqrt{1 - \nu^2} \sqrt{1 - \nu'^2} \cos \phi.
\]

Now then, if one makes calculations for the range of \( p, q, q' \), one finds that \( p \gg q, q' \) since \( p \approx 10^7 \text{ cm}^{-1} \) and \( q, q' \approx k_B T/\hbar \approx 10^5 \text{ cm}^{-1} \). The arguments of the \( \delta \)-functions become

\[
\delta(\epsilon_p \pm sq \pm sq' - \epsilon_{p+q+q'}) 
= \delta(\pm sq \pm sq' - \frac{pq\nu}{m}). \] (19)

The \( \delta \)-function integration can be done. We arrive at an expression that is proportional to the following integral:

\[
I_1 = \frac{m}{pq} \int_{-1}^1 d\nu \frac{1}{\pm sq - \frac{q^2}{2m} - \frac{pq\nu}{m}}, \]
\[
I_2 = \frac{m}{pq} \int_{-1}^1 d\nu' \frac{1}{\pm sq' - \frac{q'^2}{2m} - \frac{pq\nu'}{m}}.
\]

\( I_1 \) and \( I_2 \) contain singularities and they cannot be integrated the way they are. One can avoid the divergence by introducing a quasiparticle damping term in the denominator \(-i2\gamma\) where \( \gamma = h/\tau \). Then

\[
I_1 = \frac{m}{pq} \int_{-1}^1 d\nu \frac{1}{\pm sq - \frac{q^2}{2m} - \frac{pq\nu}{m}} + 4\gamma^2 \]
\[
= \frac{m^2}{pq^2} \frac{\pi}{2\gamma} = I_2. \] (20)

It is clear that the formula will depend on the introduced parameter \( \gamma \). It represents the inverse lifetime of the electrons in the system. This result is completely new and somewhat unexpected. We assume \( \gamma = h/\tau \) is a constant, which is the case for scattering from impurities or boundaries of the material.

The integration over \( d^3p \) can easily be performed by making the following substitutions:

\[
d^3p = p^2 dp d\Omega = p^3 \frac{4\pi}{v_F} d\epsilon_p, \]
\[
sq = \Omega, \]
\[
sq' = \Omega'.
\]

Then the integrals over \( \epsilon \) become

\[
\int_0^\infty d\epsilon f(1 - f(\epsilon \pm \Omega \pm \Omega')) = \frac{\pm \Omega \pm \Omega'}{1 - e^{-\beta \epsilon(\pm \Omega \pm \Omega')}}. \] (21)

\[
\int_0^\infty d\epsilon f(\epsilon \pm \Omega \pm \Omega')(1 - f(\epsilon)) = \frac{\pm \Omega \pm \Omega'}{e^{\beta \epsilon(\pm \Omega \pm \Omega')} - 1}. \] (22)

Since we are interested in the low-temperature region, one can make the long-wave approximation. For metals:

\[
\]
After this last variable change, we obtain the expression
\[
\frac{dE_e}{dt} = \frac{m^2V^4(0)}{16\pi^3\gamma_0\rho(nM)^2s^3h} \frac{1}{\beta_L^2} J_1(\alpha),
\]
(24)
where
\[
J_1(\alpha) = \int_0^\infty dx \frac{x^2}{e^x-1} \int_0^\infty dx' \frac{x'^2}{e^{x'}-1} (x+x')^2
\]
\[
\times \left[ e^{(x+x')}-e^{\alpha(x+x')} \right] e^{\alpha(x+x')} - 1
\]
\[
+ 2 \int_0^\infty dx \frac{x^2}{e^x-1} \int_0^\infty dx' \frac{x'^2}{e^{x'}-1}
\]
\[
\times (x'-x)^2 \frac{e^x}{e^{\alpha(x'-x)} - 1}.
\]
(25)

Notice that if \( \alpha=1 \) or \( T_L=T_e \) then \( dE/dt=0 \). The system has reached thermal equilibrium. For the case when \( \alpha\neq0 \) one can see that the heat exchange is proportional to the lattice temperature as \( T_L^8 \). Also, the thermal rate is proportional to the lifetime of the electrons. For very long lifetimes, this energy loss rate can be larger than that due to the one phonon processes.

FIG. 2. \( J_1(\alpha) \) vs \( \alpha \) for the Compton-like scattering. The integral is defined in Eq. (25).

\[
qM^2(q) = \frac{V^2 \Omega^2}{2nM} s^s T_L^s.
\]
(23)

where \( (nM) \) is the mass density of the ion. Further substitutions can be made:
\[
x = \beta_L \Omega,
\]
\[
x' = \beta_L \Omega',
\]
\[
\alpha = \frac{\beta_L}{\beta_L}.
\]

The system reaches thermal equilibrium. For the case when \( \alpha > 0 \) one can see that the heat exchange is proportional to the lattice temperature as \( T_L^8 \). Also, the thermal rate is proportional to the lifetime of the electrons. For very long lifetimes, this energy loss rate can be larger than that due to the one phonon processes.

III. ELECTRON-DUAL-PHONON PROCESSES

We perform the same calculation for the electron-dual-phonon processes that arise from the second-order term of the interaction Hamiltonian:
\[
H_{int} = -\frac{1}{2} \sum_{q,q'} \sum_p M(q,q') C^{+}_q C^+_q, C^+_p C^{+}_p (a_i + a^{+}_q)
\]
\[
\times (a_{q'} + a^{+}_{-q'}),
\]
(26)
where
\[
M(q,q') = \frac{1}{(4(nM)^2s^3h)^{1/2}} V(|q+q'|)\hat{\epsilon}_{q'} \cdot \hat{\epsilon}_q
\]
\[
\times ((q+q') \cdot \hat{\epsilon}_{q'}).
\]
(27)

There is only one such process and the Feynman diagram is given in Fig. 3.

Again we use Matsubara’s Green’s function method.\(^6\) The self-energy of an electron is expressed as follows:
\[
\Sigma(p,i\omega) = \frac{1}{\beta^2} \sum_{i\omega} \sum_{i\omega'} \int \frac{d^3q d^3q'}{(2\pi)^3} M^2(q,q') D^0(q,i\omega)
\]
\[
\times D^0(q',i\omega') G^0(p+q+q',ip+i\omega+i\omega').
\]
(28)
One can do the summations over \( i \omega \) and \( i \omega' \) and make analytic continuation from \( i \rho \) to \( \epsilon_\rho + i \delta \). The imaginary part of the self-energy is found to be as follows:

\[
\text{Im} \Sigma(p,ip) = \frac{2\pi}{2\hbar} \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3q'}{(2\pi)^3} \frac{M^2(q,q') \cdot 1}{M^2(q,q') \cdot 1}.
\]  

(29)

where

\[
I = \{ \delta(\epsilon + \omega + \omega' - \xi) [NN'f(1-f') - f'(1-f)(N+1) \\
\times (N'+1)] + \delta(\epsilon - \omega - \omega' - \xi') [N(N+1)(N'+1) \\
\times f(1-f') - f'(1-f)NN'] + \delta(\epsilon - \omega + \omega' - \xi') \\
\times [(N+1)N'f(1-f') - f'(1-f)N(N'+1)] \\
+ \delta(\epsilon + \omega - \omega' - \xi') [N(N'+1)f(1-f') \\
- f'(1-f)(N+1)N'] \}. 
\]  

(30)

For the rate of energy exchange we use the same formula as before:

\[
\frac{dE}{dt} = \frac{2\pi}{2\hbar} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3q'}{(2\pi)^3} \frac{M^2(q,q') \cdot 1}{M^2(q,q') \cdot 1}.
\]  

(31)

Notice that the expression is similar to the Compton-like scattering process but lacks the singular energy denominator terms. Here the final result will not depend on any quasiparticle damping. The actual integration for this type of processes is simpler than the one for the Compton-like scattering. We make the same approximations and assumptions as before: (i) we consider only acoustic phonons with spectrum \( \omega = sq \) and electrons with spectrum \( \epsilon = p^2/2m \); (ii) the temperatures for the electrons and the lattice are different and depend on time; (iii) \( p \gg q, q' \); (iv) we take the long-wave approximation—\( q \) along the unit vector \( \hat{e}_q \) and \( q' \) along the other unit vector \( \hat{e}_{q'} \).

To perform the integration in this case one can do the following substitutions:

\[
q + q' = Q, \\
\rho \cdot \hat{Q} = \nu. 
\]  

(32)

(33)

We concentrate on the integration over \( p \) first:

\[
\int d^3p = 2\pi \int p^2dp \int d\nu. 
\]  

(34)

One can make further substitutions:

\[
(p + Q)^2 = (k)^2, \\
d
\nu = \frac{k}{pQ} dk. 
\]

In this way the integrals over \( p \) and \( k \) become integrals over \( \epsilon_\rho \) and \( \epsilon_k \) simply by noting that

\[
\epsilon_\rho = \frac{p^2}{2m}, \\
\epsilon_k = \frac{k^2}{2m}. 
\]

The \( \delta \) functions help us to carry out the integration over \( \epsilon_k \). Thus, we arrive at the same integrals over \( \epsilon_\rho \) as the ones for the Compton-like scattering, Eqs. (21) and (22).

Now then, one can make a further substitution:

\[
\hat{q} \cdot \hat{q}' = \nu'. 
\]  

(35)

Then the matrix element for the electron-dual-phonon scattering is

\[
M = \left( \frac{V(q + q')}{4(nM)^2} \right)^{1/2} [qq' + (q^2 + q'^2) \nu' + qq' \nu'^2], 
\]  

(36)

The integrals over \( q \) and \( q' \) become

\[
\int d^3q \int d^3q' = 2\pi^2 \int_0^\infty q^2dq \int_0^\infty q'^2dq' \int^1_{-1} d\nu'. 
\]

(37)

Thus, one can perform the integration over \( \nu' \). The calculation is not difficult, but long and tedious. We have to take account also of \( q \) being less or greater than \( q' \). Notice that \( Q = (q^2 + q'^2 + 2qq')^{1/2} \) is simply the length of the vector \( (q + q') \). Again we choose

\[
\Omega = sq, \\
\Omega' = sq', \\
x = \beta_L Q, \\
x' = \beta_L Q', \\
\alpha = \frac{\beta_e}{\beta_L}. 
\]

Using all of the above approximations and substitutions, the result for the heat exchange is given as follows:

\[
\frac{dE_e}{dt} = \frac{m^2V^2(0)}{32\pi^3\hbar s^2(nM)^2} \frac{1}{\beta_L} J_2(\alpha), 
\]  

(38)

where

\[
J_2(\alpha) = \left( \int_0^\infty dx \frac{x}{e^{x-1}} \int_0^x dx' \frac{x'}{e^{x'-1}} (x+x')^2 \right) \\
\times \frac{e^{x+x'} - e^{x+x'} \alpha - e^{x+x'} - e^{x+x'} - 1}{e^{x+x'} - 1} \\
+ \int_0^\infty dx \frac{x}{e^{x-1}} \int_0^x dx' \frac{x'}{e^{x'-1}} (x'-x)^2 \\
\times \frac{e^{x'} - e^x e^{x'-x}}{e^{x'-x} - 1} \left( \frac{2}{3} x^3 + \frac{16}{5} x x'^2 \\
- \frac{2}{5} x'^4 + \frac{4}{5} x x'^6 + \frac{4}{5} x x'^8 \right). 
\]  

(39)
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IV. CONCLUSION

Recent interest in quantum dot refrigeration inspired us to calculate the rate of energy exchange between electrons and phonons due to second-order processes. Such nonlinear contributions are known to be important in transport theory at low temperatures, and we thought they might be important also in heat exchange. Generally we find that the second-order processes are less important than those of the one-phonon exchange. The latter varies as $T^3$, while for the two-phonon case we find $T^8$ for the Compton processes and $T^9$ for the dual-phonon processes. The two-phonon terms decline much faster with temperature.

However, our results have an interesting feature. We found that the Compton process depends directly on the lifetime of the electrons at the Fermi surface. If this lifetime is quite long, as in pure samples, then the Compton processes could be as important as that of the one-phonon processes in limiting the heat exchange. For example, since the lifetime of the electron from scattering by phonons usually goes as $T^{-3}$ at low temperatures, then this temperature dependence reduces $T^8$ to $T^3$. Then the two-phonon process has the same temperature dependence as does the one-phonon process, as well as the same magnitude. This dependence on lifetime is the major new result in the present calculation.

For the electron-dual-phonon scattering we have obtained that the heat exchange is proportional to the lattice temperature as $T_L^9$. The temperature dependence is even stronger than the Compton processes. The result does not depend on the lifetime of the electrons. It is rather clear that for temperatures below 1 K the heat leak drops faster than that for the single-phonon scattering and for the Compton scattering.

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FIG. 4. $J_2(\alpha)$ vs ln $\alpha$ for the electron-dual-electron scattering. The integral is defined in Eq. (39).

Now, if we take again $\alpha=\beta_e/\beta_L=1$ or $T_e=T L$, then $dE_e/dt=0$, which is not surprising, the system is in thermal equilibrium. The thermal rate depends on the lattice temperature as $T_L^9$, which is stronger than the temperature dependence of the thermal rate for the Compton-like scattering processes.

A graphical dependence of the exchange integral $J_2(\alpha)$ vs the logarithm of the parameter $\alpha$ is shown in Fig. 4. Again as in the Compton processes the curve for the electron-dual-phonon process starts from 0 for $\alpha=1$ and then it rises to approximately $\alpha=2$. The graph for the electron-dual-phonon scattering is steeper than the one for the Compton-like scattering to approximately an order. However, after $\alpha=2$ the graph plateaus parallel to the x axis. The asymptotic limit is $J_2(\infty)=6874$. If $T_L>2T_e$ the amount of heat exchange in the electron-dual-phonon scattering is determined only by the lattice temperature.
