H-Spaces and Hopf Algebras

Junior Topology Seminar
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**Definition**  
An algebra $A$ over a commutative ring $K$ is a module over $K$ in which an operation of multiplication is defined satisfying, for every $a, b, c \in A$ and every $k \in K$. The distributive laws

1. (right distributivity) $a(b + c) = ab + ac$
2. (left distributivity) $(a + b)c = ac + ac$
3. (homogeneity) $k(ab) = (ka)b = a(kb)$

These three axioms are another way of saying that the binary operation $A \times A \rightarrow A$ is bilinear.
Basic Definitions

Algebras

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- Associativity: $a(bc) = a(bc)$
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We might also want to add one the following axioms to the definition

○ Associativity: $a(bc) = a(bc)$
○ A has an identity element 1 ($1a = a = a1$ for all $a$ in $A$)
Basic Definitions

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**Example** Let $M_n(K)$ be the set of all $n \times n$ matrices with entries in $K$, then $M_n(K)$ with the obvious matrix multiplication and addition is an algebra over $K$. 
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**Example** The (real) Quaternions, denoted $H$, is the four-dimensional vector space over $\mathbb{R}$ with basis denoted by $\{1, i, j, k\}$, and multiplication defined so that $1$ is the multiplicative identity element and

\[ i^2 = j^2 = k^2 = -1 \]

\[ ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j \]
Basic Definitions

Algebras

It is possible to define associativity in a way that does not explicitly refer to elements.

\[ [.,.] : A \times A \to A \]

bilinear: \[ [ax + by, z] = a[x, z] + b[y, z], \text{ and } [z, ax + by] = a[z, x] + b[z, y] \]

for all scalars \( a, b \) in \( R \) and all elements \( x, y, z \) in \( A \).

The bilinear map induces a linear map on \( \mu : A \otimes A \to A \)
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Saying that an algebra $A$ is associative is exactly as saying that the map $\mu$ has the property

$$M \circ (Id \times M) = M \circ (M \times Id) \quad (1)$$

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$$(M \circ (Id \times M))(x, y, z) = M(x, M(y, z)) = M(M(x, y), z) = (M \circ (M \times Id))(x, y, z)$$
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\[
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{M \otimes id} & A \otimes A \\
\downarrow{id \otimes M} & & \downarrow{\mu} \\
A \otimes A & \xrightarrow{\mu} & A
\end{array}
\]
Basic Definitions

Algebras

Observe that an associative, unitary \( k \)-algebra

is a pair \((A, m)\), where \( A \) is a \( k \)-module and \( m : A \otimes A \to A \) is a \( k \)-linear map, called the \textit{multiplication}, such that:

1. The following diagram is commutative:

\[
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{m \otimes id} & A \otimes A \\
\downarrow{id \otimes m} & & \downarrow{m} \\
A \otimes A & \xrightarrow{m} & A
\end{array}
\]

2. There exists a \( k \)-linear map \( u : k \to A \) such that the following diagrams commute:

\[
\begin{array}{ccc}
k \otimes A & \xrightarrow{u \otimes id} & A \otimes A \\
\downarrow & & \downarrow{id \otimes u} \\
A & & A \otimes k
\end{array}
\]

\[
\begin{array}{ccc}
& & \\
\downarrow & & \downarrow{m} \\
A & \xrightarrow{m} & A
\end{array}
\]

where the maps \( k \otimes A \to A \) and \( A \otimes k \to A \) are the canonical ones. Such a \( u \) is necessarily unique. The first of these diagrams says that the algebra \( A \) is associative and the second gives the existence of a unit \( u(1) = 1_A \) in \( A \).
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2. There is a diagonal or coproduct $\Delta : A \to A \otimes A$, a homomorphism of graded algebras satisfying $\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha + \sum_i \alpha_i' \otimes \alpha_i''$ where $|\alpha_i'| > 0$ and $|\alpha_i''| > 0$, for the $\alpha$ with $|\alpha| > 0$. 
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\[ \Delta(A_n) \subseteq \bigoplus_{i+j=-n} A_i \otimes A_j \]

The multiplication in $A \otimes A$ by the standard formula $(\alpha \otimes \beta)(\gamma \otimes \delta) = (-1)^{|\beta||\gamma|}(\alpha \gamma \otimes \beta \delta)$. 
Example The Polynomial ring $\mathbb{R}[X]$

Any polynomial $p(x) \in \mathbb{R}[x]$ is given by

$$p(x) = r_n x^n + \cdots + r_1 x^1 + r_0 x^0,$$
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$\Delta(x) \in A_0 \otimes A_1 + A_1 \otimes A_0$ and hence $\Delta(x) = x \otimes 1 + 1 \otimes x$. The fact that $\Delta$ is an algebra homomorphism determines $\Delta$ completely.
Hopf Algebras

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Comultiplication can be defined as,

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\Delta(s_n) = \sum_{i=0}^{n} s_i \otimes s_{n-i}
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For example,

$$\Delta(s_1) = 1 \otimes s_1 + s_1 \otimes 1, \quad \Delta(s_2) = 1 \otimes s_2 + s_1 \otimes s_1 + s_2 \otimes 1$$
A topological space $X$ is said to be an $H$-space (or Hopf-space) if there exists a continuous binary operation $\mu : X \times X \to X$ and a point $p \in X$ such that the functions $X \to X$ defined by $x \to \mu(p, x)$ and $x \to \mu(x, p)$ are both homotopic to the identity map via homotopies that leave $p$ fixed. The element $p$ is sometimes referred to as an ‘identity’, although it need not be an identity element in the usual sense.
**Example**  If $G$ is a topological group then it is an $H$-space. The map $\mu : X \times X \rightarrow X$ can be taken simply to be the multiplication map and the two maps $x \rightarrow \mu(p,x)$ and $x \rightarrow \mu(x,p)$ are both homotopic are homotopic because they are equal.
Examples of H-spaces

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Example  $S^1$, $S^3$, and $GL_n(\mathbb{C})$ are $H$-spaces being topological groups.
Example  If we identify $\mathbb{C}^\infty \setminus \{0\}$ with the space of polynomials by assigning the nonzero polynomial $a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ to the point $(a_0, a_1, a_2, \ldots, a_n, 0, 0, \ldots) \in \mathbb{C}^\infty \setminus \{0\}$. Multiplication of induces multiplication $\mathbb{C}^\infty \setminus \{0\}$. Identifying proportional polynomials, we obtain a multiplication in the $\mathbb{CP}^\infty$ space. $\mathbb{CP}^\infty$ is an $H$-space with the multiplication map just defined (continuous since the multiplication defined on $\mathbb{C}^\infty \setminus \{0\}$ is continuous) and the element $[1 : 0 : 0 : \ldots]$ is the identity element.
The octonions can be thought of as 8–tuples of real numbers. Every octonion is a real linear combination of the unit octonions \(\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}\) where \(e_0\) is the scalar element. As a topological space the octonions is just \(\mathbb{R}^8\) with the usual topology. It has a nonassociative, noncommutative, unital normed division \(\mathbb{R}\)–algebra structure: the algebra generated by the elements \(\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}\) and satisfy some relations.
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**Example** \( S^7 \) can be described as the set of Cayley number numbers with unit norm. Multiplication of Cayley numbers induces a nonassociative multiplication on \( S^7 \). This space is an \( H \)–space with the identity element \( (1, 0, 0, 0, 0, 0, 0, 0) \).
Examples of H-spaces

Example \( \text{Let } X \text{ be a topological space and let } X^k \text{ the product of } k \text{ copies of } X. \text{ Define the space } J(X) \text{ to be the quotient space of the disjoint union } \bigsqcup_{k \geq 1} X^k \)
Examples of H-spaces

**Example** Let $X$ be a topological space and let $X^k$ the product of $k$ copies of $X$. Define the space $J(X)$ to be the quotient space of the disjoint union $\coprod_{k \geq 1} X^k$ by identifying $(x_1, \ldots, x_i, \ldots, x_k)$ with $(\hat{x}_1, \ldots, \hat{x}_i, \ldots, x_k)$ if $x_i = e$, a chosen of $X$. We can think of $J(X)$ as $k$-tuples $(x_1, \ldots, x_k)$, $k \geq 0$ with no $x_i = e$.

This space can be given an $H$-space structure as follows. Define multiplication on $J(X)$ by $(x_1, \ldots, x_k)(y_1, \ldots, y_k) = (x_1, \ldots, x_k, y_1, \ldots, y_k)$. This multiplication is obviously associative and the identity element is $(e)$ where $e$ is element identity in $X$. To see that this is Notice that $(x_1, \ldots, x_k)(e) = (x_1, \ldots, x_k, e) = (x_1, \ldots, x_k)$ by the definition of $J(X)$. 


Let $R$ be a commutative ring. We can regard the ring $H^*(X; R) = \oplus_{n \geq 0} H^n(X; R)$ as a graded algebra over $R$. Let $X$ be a connected $H$-space. Give $X$ a CW structure and suppose it satisfies the following conditions:
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1. $X$ is path-connected, hence $H^0(X; R) = R$. 

The Cohomology algebra of an H-space is a Hopf algebra

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1. $X$ is path-connected, hence $H^0(X; R) = R$.
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$X$, being an $H$-Space, has a continuous multiplication map say $\mu : X \times X \rightarrow X$ and hence it induces an algebra homomorphism $\mu^* : H^*(X; R) \rightarrow H^*(X \times X; R)$. The algebra homomorphism $\Delta : H^*(X; R) \rightarrow H^*(X; R) \otimes_R H^*(X; R)$ is defined to be the composition $\mu^* \circ c$. 
The Cohomology algebra of an H-space is a Hopf algebra

Introduction

Want to show that $\Delta$ is actually a diagonal. Notice the following in the following commutative diagram:

$$
\begin{array}{cccc}
H^*(X;R) & \\ & \mu^* \downarrow & \searrow \\downarrow \Delta & H^*(X \times X;R) \\
& & i^* \downarrow & H^*(X;R) \\
H^*(X;R) \otimes_R H^*(X;R) & \approx & \times & H^*(X;R) \otimes_R H^*(e;R)
\end{array}
$$

(1) $P$ is defined by $\alpha^p \otimes 1 \rightarrow \alpha^p$ and $\alpha^p \otimes \alpha^q \rightarrow 0$ for $q > 0$.
(2) Since $\mu \circ i \simeq 1$ then $i^* \circ \mu^* = 1 = P \circ \Delta$.

This shows that in particular that the component of $\Delta(\alpha)$ in $H^0(X;R) \otimes H^n(X;R)$ is $\alpha \otimes 1$
(1) There is an identity element $1 \in A^0$ such that the map $R \to A^0$, $r \mapsto r \cdot 1$, is an isomorphism. In this case one says $A$ is connected.

(2) There is a **diagonal** or **coproduct** $\Delta: A \to A \otimes A$, a homomorphism of graded algebras satisfying $\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha + \sum_i \alpha_i' \otimes \alpha_i''$ where $|\alpha_i'| > 0$ and $|\alpha_i''| > 0$, for all $\alpha$ with $|\alpha| > 0$. 

The Cohomology algebra of an H-space is a Hopf algebra

Dual Hopf Algebra
A Product structure on homology of H-spaces

Pontryagin product

In $H$-space we have product structure also on homology. We had the following cross product on homology.

$$H_i(X; R) \times H_j(Y; R) \xrightarrow{\times} H_{i+j}(X \times Y; R)$$
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Since $X$ is an $H$-space then we have the map $\mu : X \times X \to X$ which induces a homomorphism on homology $\mu_* : H_*(X \times X; R) \to H_*(X, R)$
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$$H_*(X;R) \otimes H_*(X;R) \xrightarrow{\times} H_*(X \times X;R) \xrightarrow{\mu_*} H_*(X;R)$$

It is associative/commutative if $\mu$ is associative/commutative.
Suppose that $X$ is an $H$-space and consider $H_n(X; R)$ with coefficients in the field $R$. Assume that $H_n(X; R)$ is finite-dimensional for all $n$. By the exactness of
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$$0 \rightarrow \text{Ext}_R(H_{n-1}(X), R) \rightarrow H^n(X; R) \rightarrow \text{Hom}_R(H_n(X), R) \rightarrow 0$$

we obtain

$$H^n(X; R) = \text{Hom}_R(H_n(X; R), R)$$
We have the Pontryagin product

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\[ H_*(X; R) \otimes H_*(X; R) \rightarrow H_*(X; R) \]

If we dualize this we get

\[ \text{Hom}(H_*(X; R)) \rightarrow \text{Hom}((H_*(X; R) \otimes H_*(X; R)) \cong \text{Hom}((H_*(X; R)) \otimes \text{Hom}(H_*(X; R)) \]

Which implies that we have a homomorphism

\[ H^*(X; R)) \rightarrow H^*(X; R) \otimes H^*(X; R) \]

Since both Pontryagin product and the coproduct map \( \Delta \) induced by the \( H \)-space product, the coproduct \( \Delta \) and Pontryagin product are dual to each other.
What is the relation between Cohomology and Homology in H-spaces?
What is the relation between Cohomology and Homology in H-spaces?

They are dual Hopf algebras
Theorem  Let $A$ be a Hopf algebra over $R$ that is finitely generated free $R$-module in each dimension. Then the product $\pi : A \otimes A \rightarrow A$ and coproduct $\Delta : A \rightarrow A \otimes A$ have duals $\pi : A^* \rightarrow A^* \otimes A^*$ and $\Delta : A^* \otimes A^* \rightarrow A^*$ that give $A^*$ the structure of a Hopf algebra.
Thank you