PRACTICE SET FOR FINAL – SOLUTIONS

PROBLEM 1

1) $f(x) = \dfrac{x+1}{x^2}$

- Domain: $\boxed{x \neq 0}$ $\quad]-\infty, 0[\cup]0, +\infty[$

- $\cap$ with axes: with y-axis no intersection
  with x-axis $\dfrac{x+1}{x^2} = 0 \rightarrow \boxed{x = -1}$

- Symmetries: $f(-x) = \dfrac{-x+1}{x^2} \neq f(x)$, $\neq -f(x)$ $\quad$ **No symmetries**

- Limits & Asymptotes

$$\lim_{x \to -\infty} \frac{x+1}{x^2} = \lim_{x \to -\infty} \frac{1 + \frac{1}{x}}{x} = 0 \qquad \boxed{y = 0 \text{ HOR AS AT } -\infty}$$

$$\lim_{x \to +\infty} \frac{x+1}{x^2} = \lim_{x \to +\infty} \frac{1 + \frac{1}{x}}{x} = 0 \qquad \boxed{y = 0 \text{ HOR AS AT } +\infty}$$

$$\lim_{x \to 0^-} \frac{x+1}{x^2} = +\infty \qquad \boxed{x = 0 \text{ VERT AS}}$$

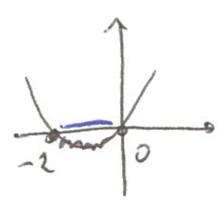
$$\lim_{x \to 0^+} \frac{x+1}{x^2} = +\infty$$

- Monotonicity

$$f'(x) = \frac{x^2 - (x+1)2x}{x^4} = -\frac{x^2+2x}{x^4}$$

crit. points $f'(x) = 0$: $x^2 + 2x = 0 \rightarrow \boxed{x = -2}$ $\quad$ $x = 0$ not acceptable

$f'(x) > 0 \quad -\dfrac{x^2+2x}{x^4} > 0 \rightarrow \dfrac{x^2+2x}{x^4} < 0 \rightarrow x^2 + 2x < 0$ $\quad \boxed{-2 < x < 0}$



		-2		0	
f'	$-$		$+$		$-$
f	$\searrow$	loc MIN	$\nearrow$		$\searrow$

local MIN $M = (-2, -\frac{1}{4})$

- Concavity $\quad f'(x) = -\frac{x+2}{x^3}$

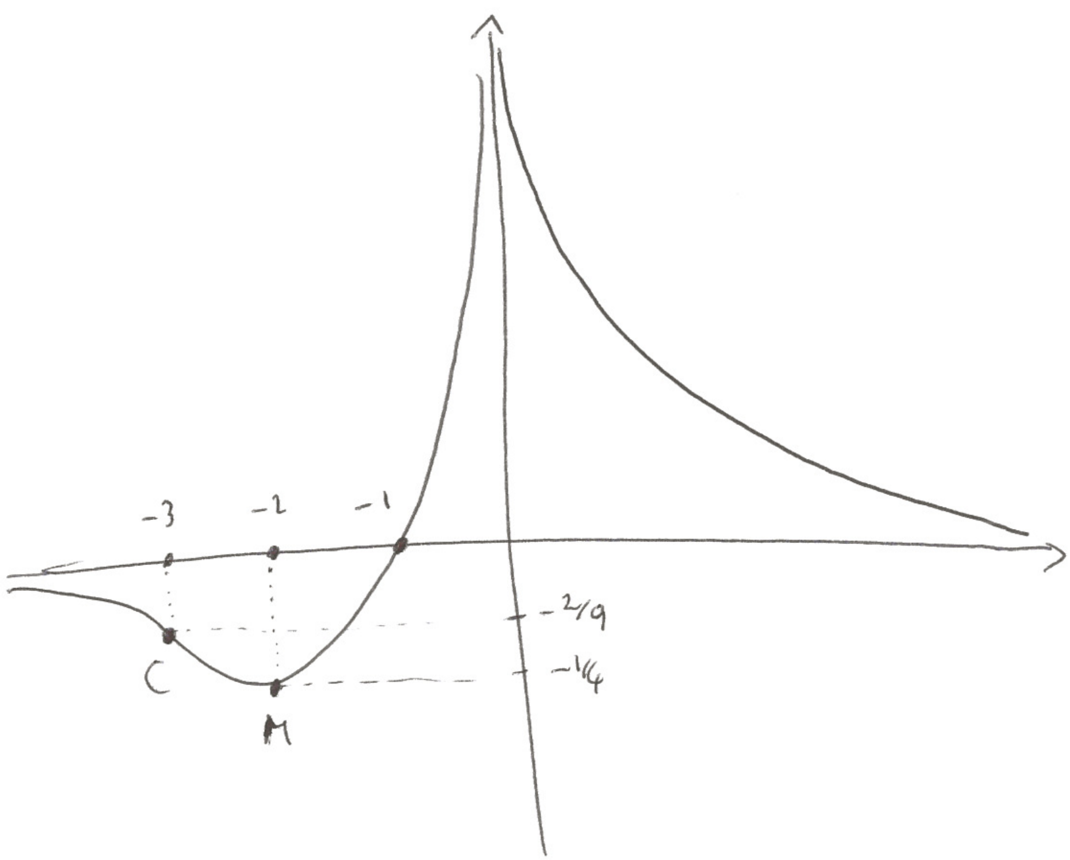
$$f''(x) = -\frac{x^3 - 3x^2(x+2)}{x^6} = -\frac{x^3 - 3x^3 - 6x^2}{x^6} = \frac{2x^3+6x^2}{x^6} = \frac{2x+6}{x^4}$$

$f''(x) = 0 \quad \boxed{x = -3}$

$f''(x) > 0 \quad \frac{2x+6}{x^4} > 0 \rightarrow 2x+6 > 0 \rightarrow x > -3$

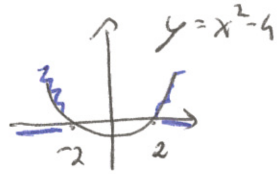
		-3		0	
f''	$-$		$+$		$+$
f	$\frown$		$\smile$		$\smile$
		INFL POINT			

inflection point $C = (-3, -\frac{2}{9})$



2) $f(x) = \ln(x^2 - 4)$

- Domain: $x^2 - 4 > 0$



$\left(\begin{array}{l} x^2 - 4 = 0 \\ x = \pm 2 \end{array} \right)$

$\boxed{x < -2 \vee x > 2}$ $\qquad$ $\boxed{]-\infty, -2[\cup]2, +\infty[}$

- ∩ with axes: x-axis: $\ln(x^2-4) = 0$

$x^2 - 4 = 1$

$\boxed{x = \pm\sqrt{5}}$ acceptable

y-axis: no intersections ($x = 0$ not in domain)

- Symmetries: $f(-x) = \ln((-x)^2 - 4) = \ln(x^2-4) = f(x)$

$\boxed{f \text{ is EVEN}}$

- Limits & Asymptotes

$\lim\limits_{x \to +\infty} \ln(x^2-4) = +\infty$ since $\lim\limits_{x \to +\infty} (x^2-4) = +\infty$

$\lim\limits_{x \to -\infty} \ln(x^2-4) = +\infty$ " $\boxed{\text{NO HOR ASYMPTOTES}}$

$\lim\limits_{x \to 2^+} \ln(x^2-4) = -\infty$ since $\lim\limits_{x \to 2^+} x^2 - 4 = 0^+$ $\to$ $\boxed{x = 2 \text{ VERT AS}}$

$\lim\limits_{x \to -2^-} \ln(x^2-4) = -\infty$ " $\to$ $\boxed{x = -2 \text{ VERT AS}}$

- Monotonicity $f'(x) = \dfrac{2x}{x^2-4}$ $\qquad$ $f'(x) = 0 \to x = 0$ not acceptable.

$f'(x) > 0$ $\qquad$ $\dfrac{2x}{x^2-4} > 0$ $\qquad$ In the domain $x^2 - 4 > 0$ so it doesn't contribute to the sign of the fraction.

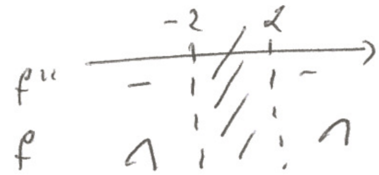
$\hookrightarrow x > 0$

	$-\infty$	-2		0		2	$+\infty$
f'	$-$	¦	//////		//////	¦	$+$
f	↘	¦	//////		//////	¦	↗

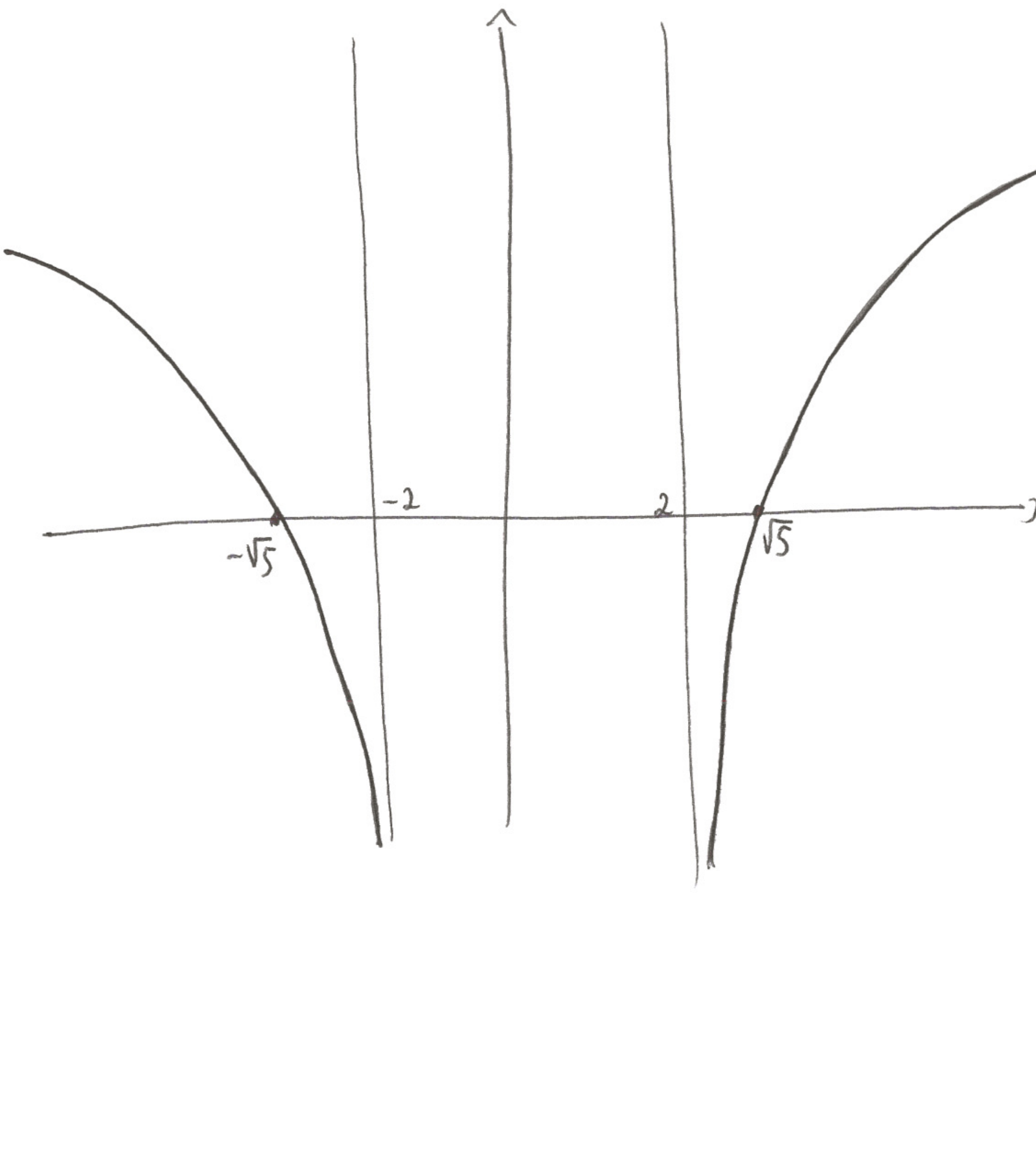
No loc max/min.

- Concavity $\quad f''(x) = \dfrac{2(x^2-4) - 2x \cdot 2x}{(x^2-4)^2} = \dfrac{-2x^2-8}{(x^2-4)^2} = -2\,\dfrac{x^2+4}{(x^2-4)^2}$

$f''(x) < 0 \quad \forall x \in$ Domain



no infl. points



PROBLEM 2

3) $\lim_{x\to+\infty} \frac{\sqrt{4+x^2}}{9x} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x\to+\infty} \frac{\cancel{x}\sqrt{\frac{4}{x^2}+1}}{9\cancel{x}} = \lim_{x\to+\infty} \frac{\sqrt{\frac{4}{x^2}+1}}{9} = \frac{1}{9}$

4) $\lim_{x\to 3} \frac{x^3-3x^2-2x+6}{x^2-3x}$ "$\frac{0}{0}$"

$x^3-3x^2-2x+6 = x^2(x-3)-2(x-3)$
$= (x-3)(x^2-2)$

$= \lim_{x\to 3} \frac{\cancel{(x-3)}(x^2-2)}{x\cancel{(x-3)}} = \frac{9-2}{3} = \frac{7}{3}$

5) $\lim_{x\to+\infty} \frac{\cos(x^3)}{x^2}$ "$\frac{?}{\infty}$"

$-\frac{1}{x^2} \le \frac{\cos(x^3)}{x^2} \le \frac{1}{x^2} \quad \forall x \neq 0$

$-\frac{1}{x^2} \to 0$, $\frac{1}{x^2} \to 0$ as $x \to +\infty$

so by the squeeze theorem $\lim_{x\to+\infty} \frac{\cos(x^3)}{x^2} = 0$

6) $\lim_{x\to+\infty} \ln(x) - x$ "$\infty - \infty$"

$= \lim_{x\to+\infty} \underset{\downarrow\, +\infty}{x} \Big(\underbrace{\frac{\ln(x)}{x} - 1}_{\downarrow\, -1} \Big)$

$\lim_{x\to+\infty} \frac{\ln(x)}{x} \overset{L'H}{=} \lim_{x\to+\infty} \frac{\frac{1}{x}}{1} = 0$

$= -\infty$

7) $\lim_{x\to 0} \frac{\cos(3x)-1}{\sin(7x)}$ "$\frac{0}{0}$"

L'Hôp: $\lim_{x\to 0} \frac{-3\sin(3x)}{7\cos(7x)} = 0$ so $\lim_{x\to 0} \frac{\cos(3x)-1}{\sin(7x)} = 0$

8) $\lim_{x\to+\infty} x \, tg\left(\frac{1}{x}\right)$ "$\infty \cdot 0$"

$= \lim_{x\to+\infty} \frac{tg\left(\frac{1}{x}\right)}{\frac{1}{x}}$ "$\frac{0}{0}$" L'Hôp: $\lim_{x\to+\infty} \frac{\sec^2\left(\frac{1}{x}\right)\left(\cancel{-\frac{1}{x^2}}\right)}{\cancel{-\frac{1}{x^2}}} = \sec^2(0) = 1$

so $\lim_{x\to+\infty} x \, tg\left(\frac{1}{x}\right) = 1$.

9) $\lim\limits_{x\to+\infty}\left(1-\frac{2}{3x}\right)^{x}$ "1^{∞}"

$$= \lim_{x\to+\infty} e^{\ln\left(1-\frac{2}{3x}\right)^{x}} = \lim_{x\to+\infty} e^{\boxed{x\ln\left(1-\frac{2}{3x}\right)}}$$

Consider $\lim\limits_{x\to+\infty} \boxed{x\ln\left(1-\frac{2}{3x}\right)}$ "$\infty\cdot 0$"

$$= \lim_{x\to+\infty} \frac{\ln\left(1-\frac{2}{3x}\right)}{\frac{1}{x}} \quad \text{"}\tfrac{0}{0}\text{"}$$

Use L'Hôp: $\lim\limits_{x\to+\infty} \dfrac{\frac{1}{1-\frac{2}{3x}}\cdot\left(\frac{2}{3\cancel{x^2}}\right)}{-\frac{1}{\cancel{x^2}}} = \lim\limits_{x\to+\infty} -\dfrac{1}{1-\frac{2}{3x}}\cdot\dfrac{2}{3} = \boxed{-\dfrac{2}{3}}$

Hence $\lim\limits_{x\to+\infty} x\ln\left(1-\frac{2}{3x}\right) = -\frac{2}{3}$ and $\lim\limits_{x\to+\infty}\left(1-\frac{2}{3x}\right)^{x} = e^{-2/3}$

10) $\lim\limits_{x\to0^+}\left(\sqrt{x}\right)^{3x}$ "0^{0}"

$$= \lim_{x\to0^+} e^{\ln\left(\sqrt{x}^{3x}\right)} = \lim_{x\to0^+} e^{3x\ln\sqrt{x}} = \lim_{x\to0^+} e^{\boxed{\frac{3}{2}x\ln(x)}}$$

Consider $\lim\limits_{x\to0^+} \frac{3}{2}x\ln(x)$ "$0\cdot\infty$"

$$= \lim_{x\to0^+} \frac{3}{2}\,\frac{\ln(x)}{\frac{1}{x}} \quad \text{"}\tfrac{\infty}{\infty}\text{"}$$

Use L'Hôp: $\lim\limits_{x\to0^+} \dfrac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to0^+}(-x) = \boxed{0}$

Hence $\lim\limits_{x\to0^+} \frac{3}{2}x\ln(x) = 0$ and $\lim\limits_{x\to0^+}\left(\sqrt{x}\right)^{3x} = e^{0} = 1$.

11) $\lim\limits_{x \to 0} \dfrac{\int_0^{3x} e^t \cos(2t)\, dt}{tg(6x)}$ "$\frac{0}{0}$"

Use L'Hôp: $\left(\int_0^{3x} e^t \cos(2t)\, dt\right)' = e^{3x} \cos(6x) \cdot 3$ since $e^t \cos(2t)$ is continuous and by the FTC plus chain rule.

$(tg(6x))' = \sec^2(6x) \cdot 6$

So $\lim\limits_{x \to 0} \dfrac{e^{3x} \cos(6x) \cdot 3}{\sec^2(6x) \cdot 6} = \dfrac{3}{6} = \frac{1}{2}$

So by L'Hôp $\lim\limits_{x \to 0} \dfrac{\int_0^{3x} e^t \cos(2t)\, dt}{tg(6x)} = \dfrac{1}{2}$

12) $\lim\limits_{x \to 0} \dfrac{\int_0^{x^4} e^{t^2}\, dt}{\int_0^{x^2} e^t t\, dt}$ "$\frac{0}{0}$"

Use L'Hôp: $\left(\int_0^{x^4} e^{t^2} dt\right)' = e^{(x^4)^2} \cdot 4x^3$ (since e^{t^2} is cont and by the FTC plus chain rule.)

$\left(\int_0^{x^2} e^t \cdot t\, dt\right)' = e^{x^2} \cdot x^2 \cdot 2x$ for the same reason.

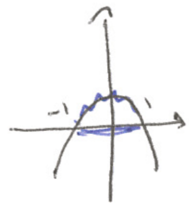
So $\lim\limits_{x \to 0} \dfrac{e^{x^8}\, 4\cancel{x^3}}{e^{x^2}\, 2\cancel{x^3}} = \dfrac{4}{2} = 2$

Hence by L'Hôp. $\lim\limits_{x \to 0} \dfrac{\int_0^{x^4} e^{t^2}\, dt}{\int_0^{x^2} e^t t\, dt} = 2$

PROBLEM 3

13) $\sqrt{1-x^2} = x$

Domain: $1-x^2 \geq 0$



$\boxed{-1 \leq x \leq 1}$

Before squaring check positivity : $\boxed{x \geq 0}$

Putting domain and positivity together : $\boxed{0 \leq x \leq 1}$

$1-x^2 = x^2$

$2x^2 = 1 \rightarrow x = \pm\sqrt{\frac{1}{2}}$ only $\boxed{x = \sqrt{\tfrac{1}{2}} \text{ acceptable}}$

(Indeed for $x = -\sqrt{\frac{1}{2}}$ we would get $\sqrt{1-\frac{1}{2}} = -\sqrt{\frac{1}{2}}$

$\sqrt{\frac{1}{2}} = -\sqrt{\frac{1}{2}}$ which is false)

14) $x^5 - x = 2$

$x^5 - x - 2 = 0$

Consider $f(x) = x^5 - x - 2$ it is continuous

$f(0) = -2 < 0$

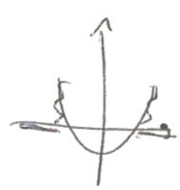
$f(2) = 2^5 - 4 > 0$

So by the IVT there exists at least one solution in the interval $]0,2[$.

$f'(x) = 5x^4 - 1$ $\quad f'(x) = 0$ $\quad x^4 = \frac{1}{5}$ $\quad x = \pm\frac{1}{\sqrt[4]{5}}$

$f'(x) > 0$ $\quad 5x^4 - 1 = (\sqrt{5}x^2 - 1)\underbrace{(\sqrt{5}x^2+1)}_{\text{always} > 0} > 0$

So $\sqrt{5}x^2 - 1 > 0$



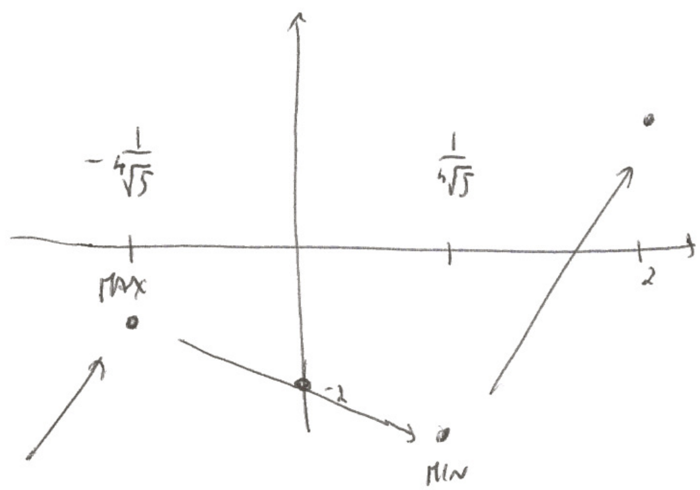
$\boxed{x < -\frac{1}{\sqrt[4]{5}} \vee x > \frac{1}{\sqrt[4]{5}}}$

		$-\frac{1}{\sqrt[4]{5}}$		$\frac{1}{\sqrt[4]{5}}$	
f'	$+$		$-$		$+$
f	$\nearrow$		$\searrow$		$\nearrow$
		loc max		loc min	

$$f\left(\frac{1}{\sqrt[4]{5}}\right) = \frac{1}{(\sqrt[4]{5})^5} - \frac{1}{\sqrt[4]{5}} - 2 = \frac{1}{5\sqrt[4]{5}} - \frac{1}{\sqrt[4]{5}} - 2 = \frac{-4-10\sqrt[4]{5}}{5\sqrt[4]{5}} < 0$$

$$f\left(-\frac{1}{\sqrt[4]{5}}\right) = -\frac{1}{5\sqrt[4]{5}} + \frac{1}{\sqrt[4]{5}} - 2 = \frac{-1+5-10\sqrt[4]{5}}{5\sqrt[4]{5}} < 0$$

hence the behaviour of the function is :

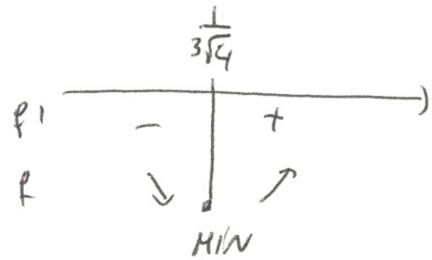


So there is exactly 1 solution.

15) $\underbrace{x^4 - x + 2}_{f(x)} = 0$

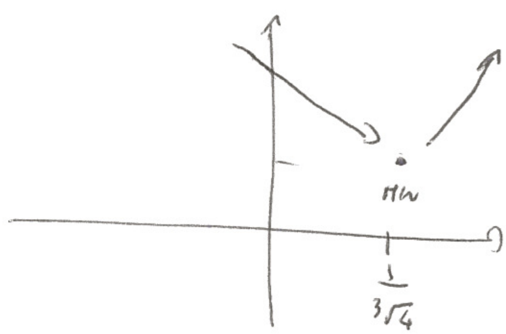
$f'(x) = 4x^3 - 1$ $\qquad f'(x) = 0 \to x = \frac{1}{\sqrt[3]{4}}$

$f'(x) > 0 \quad 4x^3 - 1 > 0 \to x > \frac{1}{\sqrt[3]{4}}$



$f\left(\frac{1}{\sqrt[3]{4}}\right) = \frac{1}{(\sqrt[3]{4})^4} - \frac{1}{\sqrt[3]{4}} + 2 = \frac{1}{4\sqrt[3]{4}} - \frac{1}{\sqrt[3]{4}} + 2 = \frac{1-4+8\sqrt[3]{4}}{4\sqrt[3]{4}} > 0$

so f has a positive absolute min, hence the equation $f(x) = 0$ has no real solutions



PROBLEM 4

16) $f(x) = \dfrac{\operatorname{arctg}(x^2)}{e^{-x}} = e^x \operatorname{arctg}(x^2)$

$f'(x) = e^x \operatorname{arctg}(x^2) + e^x \dfrac{1}{1+x^4} \cdot 2x$

17) $f(x) = \sqrt[4]{\ln(x)} = (\ln(x))^{1/4}$

$f'(x) = \dfrac{1}{4} \ln(x)^{-3/4} \cdot \dfrac{1}{x}$

18) $f(x) = \arcsin\left(\dfrac{3x}{x+1}\right)$

$f'(x) = \dfrac{1}{\sqrt{1-\left(\frac{3x}{x+1}\right)^2}} \cdot \dfrac{3(x+1)-3x}{(x+1)^2}$

19) $f(x) = tg^5(3x^2) = (tg(3x^2))^5$

$f'(x) = 5\,(tg(3x^2))^4 \cdot \sec^2(3x^2) \cdot 6x$

20) $f(x) = (x-2)^{\sin(x)} = e^{\ln((x-2)^{\sin x})} = e^{\sin(x) \cdot \ln(x-2)}$

$f'(x) = e^{\sin(x)\ln(x-2)} \left(\cos(x) \ln(x-2) + \dfrac{\sin(x)}{x-2} \right)$

21) $f(x) = \int_1^x |\sin(t)|\, dt$

Since $|\sin(t)|$ is a continuous function, by the FTC

$f'(x) = |\sin(x)|$

22) $f(x) = \int_0^{x^2+x} \frac{\operatorname{arctg}(t)}{1+t^2}\, dt$

Since $1+t^2 \neq 0 \ \forall t$, $\frac{\operatorname{arctg}(t)}{1+t^2}$ is continuous everywhere

Hence by the FTC and the chain rule we get

$$f'(x) = \frac{\operatorname{arctg}(x^2+x)}{1+(x^2+x)^2} \cdot (2x+1)$$

23) $f(x) = \int_{-x}^{x^2} e^{t^2} t^5\, dt = \int_{-x}^{0} e^{t^2} t^5\, dt + \int_0^{x^2} e^{t^2} t^5\, dt$

$$= -\int_0^{-x} e^{t^2} t^5\, dt + \int_0^{x^2} e^{t^2} t^5\, dt$$

Since $e^{t^2} t^5$ is continuous, by the FTC and the chain rule

$$f'(x) = -e^{(-x)^2} (-x)^5 (-1) + e^{(x^2)^2} (x^2)^5 \cdot 2x$$

$$= -e^{x^2} x^5 + e^{x^4} x^{10} \cdot 2x.$$

PROBLEM 5

A) $f(x) = \operatorname{arctg}(x)$ Find tg line at $x=1$.

point : $(1, \operatorname{arctg}(1)) = (1, \pi/4)$

slope : $f'(x) = \frac{1}{1+x^2}$ $f'(1) = \frac{1}{2}$

eq of tg : $\boxed{y - \pi/4 = \frac{1}{2}(x-1)}$

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points where tg is horizontal : $f'(x) = 0$

$\frac{1}{1+x^2} = 0$ $\nexists x \in \mathbb{R}$.

B) $x^3 - \sin(y) + xy = 8$ tg line at $(2,0)$

differentiate w/t x : $3x^2 - \cos(y)\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$

$$\frac{dy}{dx}(x - \cos(y)) = -y - 3x^2$$

$$\frac{dy}{dx} = -\frac{y+3x^2}{x-\cos(y)}$$

at $(2,0)$ $\frac{dy}{dx} = -\frac{12}{2-1} = -12$

eq of tg line at $(2,0)$: $\boxed{y = -12(x-2)}$

*) Are there any points where the tg is horizontal?

$$\begin{cases} \frac{dy}{dx} = 0 \rightsquigarrow \frac{y+3x^2}{x-\cos(y)} = 0 \rightarrow y = -3x^2 & (\text{slope} = 0) \\ x^3 - \sin(y) + xy = 8 & (\text{points belong to curve}) \end{cases}$$

there are points if the system $\begin{cases} y = -3x^2 \\ x^3 - \sin(y) + xy = 8 \end{cases}$ has solutions.

$$x^3 - \sin(-3x^2) + x(-3x^2) - 8 = 0$$

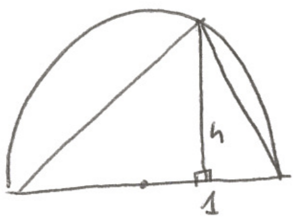
$$x^3 + \sin(3x^2) - 3x^3 - 8 = 0$$

$$\underbrace{\sin(3x^2) - 2x^3 - 8}_{g(x)} = 0$$

$\left.\begin{array}{l} g \text{ is cont} \\ g(0) = -8 < 0 \\ g(-2) = \sin(12) + 16 - 8 > 0 \end{array}\right\}$ So by the IVT there is at least one solution to the equation, hence to the system

**) Note: to be precise we should make sure that the solution doesn't satisfy $x = \cos(y)$ which is the denominator of $\frac{dy}{dx}$

c)



h: height of triangle

r = 1 radius of circle.

Find h such that Area is max.

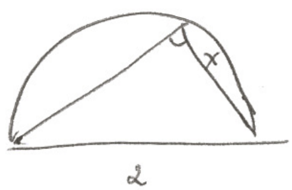
$A = \frac{2 \cdot h}{2} = h \qquad 0 < h \leq 1.$

$A' = 1$ no critical points.

$A \uparrow$, so clearly it has a max at $h = 1$.

this corresponds to an isosceles triangle.

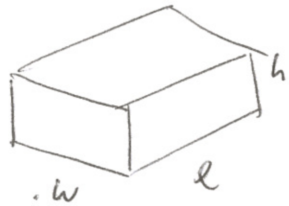
Note: if you set one side to be an unknown x then the other side is $\sqrt{4 - x^2}$ by Pythagoras and the Area function is



$$A(x) = \frac{x\sqrt{4 - x^2}}{2}$$ since the triangle is right

this leads to the same answer but the computations are more complicated.

D)



Known: $l = 3w$

cost for top/bottom : $10 \$/ft^2$

" sides : $6 \$/ft^2$

$V = 50\ ft^3$

Unknown

w, l, h s.t. cost minimized

Cost for top/bottom : $10 \cdot 2wl = 20 \cdot 3w^2 = 60w^2\ \$$

cost for sides : $6(2wh + 2lh) = 12(wh + 3wh) = 48wh\ \$$

$V = 50 \to 3w^2 h = 50 \to h = \frac{50}{3w^2}$

Tot Cost function $C(w) = 60w^2 + \frac{48 \cdot 50}{3w} = 60w^2 + \frac{16 \cdot 50}{w}$ $\boxed{w > 0}$

critical points $C'(w) = 0$

$$120w - \frac{16 \cdot 50}{w^2} = 0$$

$$w^3 = \frac{16 \cdot 5\cancel{0}}{12\cancel{0}} = \frac{4 \cdot 5}{3}$$

$$\boxed{w = \sqrt[3]{\frac{4 \cdot 5}{3}}}$$

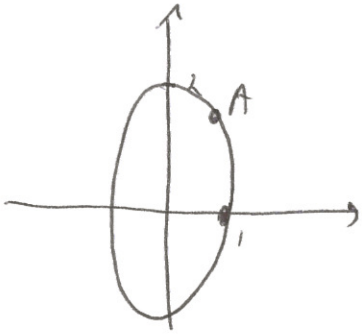
2nd deriv. test : $C''(w) = 120 + \frac{2 \cdot 16 \cdot 50}{w^3} > 0$ for $w > 0$

hence C has a min at $\boxed{w = \sqrt[3]{\frac{4 \cdot 5}{3}}}$

$\boxed{l = 3\sqrt[3]{\frac{4 \cdot 5}{3}}}$

$\boxed{h = \frac{50}{3\left(\sqrt[3]{\frac{4 \cdot 5}{3}}\right)^2}}$

E) Points on $x^2 + \frac{y^2}{4} = 1$ furthest from $P=(1,0)$



Point on the ellipse $A:(x,y)$ s.t. $y^2 = 4-4x^2$

(or $x^2 = 1 - y^2/4$)

distance function between A and P

$$d = \sqrt{(x-1)^2+y^2} = \sqrt{x^2-2x+1+4-4x^2}$$

$$d = \sqrt{-3x^2-2x+5}$$

Maximizing
Minimizing d is the same as maximizing
minimizing d^2.

$D = d^2 = -3x^2-2x+5$ $\quad -1 \le x \le 1$

$D' = -6x-2$

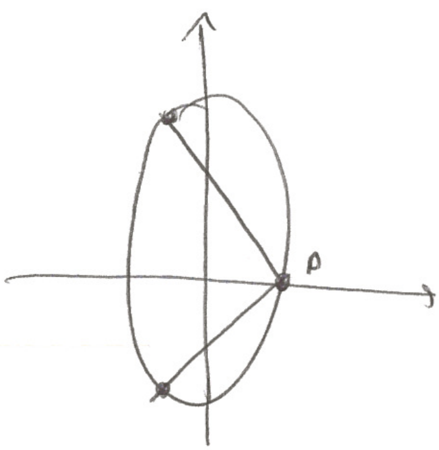
$D' = 0 \quad x = -\frac{1}{3}$

$D'' = -6 < 0$ $\quad \frown$ so max at $x = -\frac{1}{3}$

$$y^2 = 4 - \frac{4}{9} = \frac{32}{9}$$

So the points furthest from $P=(1,0)$ are

$$\left(-\frac{1}{3}, \frac{\sqrt{32}}{3}\right), \left(-\frac{1}{3}, -\frac{\sqrt{32}}{3}\right)$$



F)

V: volume

r: radius

__Known:__ $\frac{dV}{dt} = 1 \ cm^3/s$

__Unknown__ $\frac{dr}{dt} = ?$ when $V = 5 \ cm^3$.

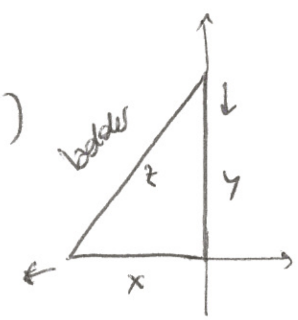
$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

when $V = 5$ $\quad \frac{4}{3}\pi r^3 = 5 \rightarrow r^3 = \frac{15}{4\pi} \rightarrow r = \sqrt[3]{\frac{15}{4\pi}}$

so $\frac{dr}{dt} = \frac{1}{4\pi\left(\sqrt[3]{\frac{15}{4\pi}}\right)^2} \cdot 1 \ cm/s$.

G)



__Known__ $\frac{dy}{dt} = -1 \ cm/s$

when $x = -10$ cm $\frac{dx}{dt} = -2 \ cm/s$

__Unknown__ $z = ?$

$$x^2 + y^2 = z^2$$

Note z is constant, so $\frac{dz}{dt} = 0$

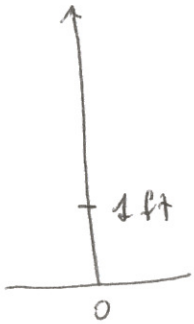
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$-10 \cdot (-2) + y(-1) = 0 \rightarrow y = 20 \ cm$$

so $z = \sqrt{x^2+y^2} = \sqrt{100+400} = \sqrt{500} = 10\sqrt{5}$ cm.

H)



$s(0) = 1$ ft

$v(0) = \frac{1}{2}$ ft/s.

max height = ?

Assume acceleration is constant

$a(t) = -g$ the acceleration of gravity.

$v(t) = \int a(t)\,dt = \int -g\,dt = -gt + C$

$v(0) = \frac{1}{2} \Rightarrow C = \frac{1}{2} \Rightarrow \boxed{v(t) = -gt + \frac{1}{2}}$

$s(t) = \int v(t)\,dt = \int \left(-gt + \frac{1}{2}\right) dt = -g\frac{t^2}{2} + \frac{1}{2}t + D$

$s(0) = 1 \rightarrow D = 1$

$$\boxed{s(t) = -g\frac{t^2}{2} + \frac{1}{2}t + 1}$$

maximize $s(t)$! $s'(t) = 0$

$-gt + \frac{1}{2} = 0 \rightarrow t = \frac{1}{2g}$

$s''(t) = -g < 0$ so s has a max at $t = \frac{1}{2g}$

and the max height is $\boxed{s\left(\frac{1}{2g}\right) = -\frac{g}{2}\left(\frac{1}{2g}\right)^2 + \frac{1}{4g} + 1}$

Find $v(0) = v_0$ s.t. max height = 50.

For $v(0) = v_0$ we get $v(t) = -gt + C$ with $C = v_0$

$v(t) = -gt + v_0$

then $s(t) = \int v(t)\,dt = -g\frac{t^2}{2} + v_0 t + D$ with $D = 1$

so $\boxed{s(t) = -g\frac{t^2}{2} + v_0 t + 1}$

$s'(t) = -gt + v_0$

$s'(t) = 0 \quad \to \quad t = \frac{v_0}{g}$ and max height reached is

$$s\left(\frac{v_0}{g}\right) = -\cancel{g}\frac{v_0^2}{2g^{\cancel{2}}} + \frac{v_0^2}{g} + 1 = \frac{v_0^2}{2g} + 1$$

Find v_0 s.t. $\frac{v_0^2}{2g} + 1 = 50 \quad \to \quad v_0^2 = 49 \cdot 2g$

$$\boxed{v_0 = 7\sqrt{2g}}$$

PROBLEM 6

24) $\int_{-1}^{1} \left(e^{-3x} - \frac{3}{1+x^2}\right) dx$

$$\int e^{-3x} - \frac{3}{1+x^2}\, dx = \frac{e^{-3x}}{-3} - 3\,\mathrm{arctg}(x) + c$$

$$\int_{-1}^{1} \left(e^{-3x} - \frac{3}{1+x^2}\right) dx = \left[\frac{e^{-3x}}{-3} - 3\,\mathrm{arctg}(x)\right]_{-1}^{1}$$

$$= \left(\frac{e^{-3}}{-3} - 3\,\mathrm{arctg}(1)\right) - \left(\frac{e^{3}}{-3} - 3\,\mathrm{arctg}(-1)\right)$$

$$= -\frac{e^{-3}}{3} - 3\pi/4 + \frac{e^{3}}{3} + 3\pi/4 = \frac{e^{3} - e^{-3}}{3} - \frac{3}{2}\pi$$

25) $\int_{0}^{1} \frac{2}{\sqrt{10-2x^2}}\, dx$

$$\int \frac{2}{\sqrt{10-2x^2}}\, dx = 2\int \frac{1}{\sqrt{10}\sqrt{1-\frac{2x^2}{10}}}\, dx = \frac{2}{\sqrt{10}} \int \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{5}}\right)^2}}\, dx$$

$$= \frac{2}{\sqrt{10}} \cdot \mathrm{arcsin}\left(\frac{x}{\sqrt{5}}\right) \cdot \sqrt{5} = \sqrt{2}\, \mathrm{arcsin}\left(\frac{x}{\sqrt{5}}\right)$$

$$\int_{0}^{1} \frac{2}{\sqrt{10-2x^2}}\, dx = \left[\sqrt{2}\, \mathrm{arcsin}\left(\frac{x}{\sqrt{5}}\right)\right]_{0}^{1} = \sqrt{2}\, \mathrm{arcsin}\left(\frac{1}{\sqrt{5}}\right) - 0$$

26) $\int x^2 \cos(2x)\,dx$ by parts twice

$$= x^2 \frac{\sin(2x)}{2} - \int x \sin(2x)\,dx$$

$$= \frac{x^2 \sin(2x)}{2} - \left[-x \frac{\cos(2x)}{2} + \int \frac{\cos(2x)}{2}\,dx \right]$$

$$= \frac{x^2 \sin(2x)}{2} + \frac{x \cos(2x)}{2} - \frac{\sin(2x)}{4} + C$$

27) $\int \underline{x} \cos(x^2)\,\underline{dx}$ substitution $u = x^2$

$du = \underline{2x\,dx} \rightarrow x\,dx = \frac{1}{2}du$

$$= \frac{1}{2}\int \cos(u)\,du$$

$$= \frac{1}{2}\sin(u) + C = \frac{1}{2}\sin(x^2) + C$$

28) $\int \frac{1}{x\sqrt{\ln(x)}}\,dx = \int \frac{(\ln(x))^{-1/2}}{x}\,dx$ $u = \ln(x)$

$du = \frac{1}{x}\,dx$

$$= \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{\ln(x)} + C.$$

29) $\int x \operatorname{arctg}(2x)\,dx$ by parts

$$= \frac{x^2}{2}\operatorname{arctg}(2x) - \int \frac{x^2}{2} \cdot \frac{1}{1+4x^2} \cdot 2\,dx$$

$$= \frac{x^2}{2}\operatorname{arctg}(2x) - \int \frac{x^2}{1+4x^2}\,dx$$

Compute separately $\int \frac{x^2}{1+4x^2}\,dx = \frac{1}{4}\int \frac{4x^2}{1+4x^2}\,dx = \frac{1}{4}\int \frac{4x^2+1-1}{1+4x^2}\,dx$

$$= \frac{1}{4}\left(\int 1\,dx - \int \frac{1}{1+(2x)^2}\,dx \right) = \frac{1}{4}\left(x - \frac{\operatorname{arctg}(2x)}{2} \right) + C$$

In conclusion

$$\int x \, arctg(2x)\, dx = \frac{x^2}{2} arctg(2x) - \frac{1}{4} x + \frac{arctg(2x)}{8} + C$$

30) $\int e^{\sin(x)} \cos(x)\, dx$

$u = \sin(x)$

$du = \cos(x)\, dx$

$= \int e^u \, du$

$= e^u + C = e^{\sin(x)} + C.$

31) $\int \sin^5(x)\, dx = \int \sin^4(x) \sin(x)\, dx = \int (1-\cos^2(x))^2 \sin(x)\, dx$

$u = \cos(x)$

$du = -\sin(x)\, dx$

$= -\int (1-u^2)^2\, du$

$= \int (-1 + 2u^2 - u^4)\, du$

$= -u + \frac{2}{3}u^3 - \frac{u^5}{5} + C$

$= -\cos(x) + \frac{2}{3}\cos^3(x) - \frac{1}{5}\cos^5(x) + C.$

32) $\int \sec^2(x)\, tg^4(x)\, dx$

$u = tg(x)$

$du = \sec^2(x)\, dx$

$= \int u^4\, du$

$= \frac{1}{5}u^5 + C = \frac{1}{5} tg^5(x) + C$

33) $\int \frac{6x+3}{x^2+x-1}\,dx = 3\int \frac{2x+1}{x^2+x-1}\,dx$ $\quad u = x^2+x-1$, $du = (2x+1)\,dx$

$= 3\int \frac{1}{u}\,du = 3\ln|u| + C = 3\ln|x^2+x-1| + C$

34) $\int \frac{\arcsin(x)}{\sqrt{1-x^2}}\,dx$ $\quad u = \arcsin(x)$, $du = \frac{1}{\sqrt{1-x^2}}\,dx$

$= \int u\,du$

$= \frac{u^2}{2} + C = \frac{\arcsin^2(x)}{2} + C$

35) $\int x e^{-2x}\,dx$ by parts

$= x\frac{e^{-2x}}{-2} + \int \frac{e^{-2x}}{+2}\,dx = -\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C$

36) $\int \frac{x^3}{1+x^8}\,dx = \int \frac{x^3}{1+(x^4)^2}\,dx$ $\quad u = x^4$, $du = 4x^3\,dx$

$= \frac{1}{4}\int \frac{1}{1+u^2}\,du = \frac{1}{4}\operatorname{arctg}(u) + C = \frac{1}{4}\operatorname{arctg}(x^4) + C$

37) $\int \frac{x^7}{1+x^8}\,dx$ $\quad u = 1+x^8$, $du = 8x^7\,dx$

$= \frac{1}{8}\int \frac{1}{u}\,du = \frac{1}{8}\ln|u| + C = \frac{1}{8}\ln(1+x^8) + C$

38) $\int \frac{1}{\sqrt{x}\,(1+\sqrt{x})}\,dx$

$u = 1+\sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx \rightarrow \frac{dx}{\sqrt{x}} = 2du$

$= \int \frac{2}{u}\,du = 2\ln|u| + c = 2\ln|1+\sqrt{x}| + c$

39) $\int \frac{1}{1+\sqrt{x}}\,dx$

$x = u^2$

$dx = 2u\,du$

$= \int \frac{1}{1+u}\,2u\,du = 2\int \frac{u+1-1}{1+u}\,du$

$= 2\left(\int 1\,du - \int \frac{1}{1+u}\,du\right)$

$= 2\left(u - \ln|1+u|\right) + c$

$= 2\left(\sqrt{x} - \ln(1+\sqrt{x})\right) + C.$