1 Problem 1

1.1 Part A

In order to show that this function has constant returns to scale, multiply each factor of production to some factor $z > 0$ to get

$$Y' = \left[ a(zK)^{-\beta} + (zL)^{-\beta} \right]^{-\frac{1}{\beta}} = \left[ z^{-\beta} (aK^{-\beta} + L^{-\beta}) \right]^{-\frac{1}{\beta}}$$

$$= z \left( aK^{-\beta} + L^{-\beta} \right)^{-\frac{1}{\beta}} = zY.$$

Since multiplying each factor of production by some positive factor $z$ increase production by a factor of exactly $z$, this production function has constant returns to scale.

1.2 Part B

The marginal product of capital is

$$\frac{\partial Y}{\partial K} = aK^{-\beta-1} \left[ aK^{-\beta} + L^{-\beta} \right]^{-\frac{1}{\beta}-1},$$

while the marginal product of labor is

$$\frac{\partial Y}{\partial L} = L^{-\beta-1} \left[ aK^{-\beta} + L^{-\beta} \right]^{-\frac{1}{\beta}-1}$$

1.3 Part C

Since capital earns its marginal product, the total returns to capital is $\frac{\partial K}{\partial Y} \cdot K$, and therefore the fraction of returns to capital over output is

$$\frac{\frac{\partial K}{\partial Y} \cdot K}{Y} = \frac{aK^{-\beta-1} \cdot K \left[ aK^{-\beta} + L^{-\beta} \right]^{-\frac{1}{\beta}-1}}{[aK^{-\beta} + L^{-\beta}]^{-\frac{1}{\beta}}} = \frac{aK^{-\beta}}{a + \left( \frac{L}{K} \right)^{-\beta}} = \frac{a}{a + k^{\beta}}$$

In order to find whether or not this falls or rises when $k$ rises, we simply take the derivative of this expression with respect to $k$ and see whether it is positive or negative.

$$\frac{\partial}{\partial k} \left( \frac{a}{a + k^{\beta}} \right) = \frac{a \beta k^{\beta-1}}{(a + k^{\beta})^2}$$
Notice that when $\beta = 0$, then the numerator is 0, and an increase in the capital-labor ratio has no effect on the fraction of total output that goes to capital. The denominator is always 0, so the derivative is negative if $\beta < 0$, and is positive if $\beta > 0$.

2 Problem 2

We have that

$$y_a = 5y_b \Rightarrow Ak_a^\alpha = 5Ak_b^\alpha \Rightarrow k_a = 5^{\frac{1}{\alpha}}k_b$$

Plugging in when $\alpha = \frac{1}{3}$ we have

$$k_a = 5^3k_b = 125k_b,$$

and when $\alpha = \frac{2}{3}$,

$$k_a = 5^2k_b \approx 11.2k_b.$$

By this we see that a seemingly small change in $\alpha$ can have a huge effect on the predictions of the model in relation to the level of capital needed to create differences in income.

We should get similar results in relation to differences in the marginal product of capital.

Case 1: $\alpha = \frac{1}{3}$

$$\text{MPK}_a = \frac{\partial y_a}{\partial k} = \alpha k_a^{\alpha-1} = \alpha \left(5^3k_b\right)^{-\frac{2}{3}} = \frac{\alpha}{25}k_b^{-\frac{2}{3}} = \frac{1}{25}\text{MPK}_b$$

Case 2: $\alpha = \frac{2}{3}$

$$\text{MPK}_a = \frac{\partial y_a}{\partial k} = \alpha k_a^{\alpha-1} = \alpha \left(5^\frac{2}{3}k_b\right)^{-\frac{1}{3}} = \frac{\alpha}{\sqrt[3]{5}}k_b^{-\frac{2}{3}} = \frac{1}{\sqrt[3]{5}}\text{MPK}_b$$

3 Problem 3

3.1 Part A

Since $\alpha = \frac{1}{2}$, we have $y = k^{\frac{1}{2}}$. Notice that if this economy is in the steady state, we must have that

$$\dot{k} = sf(k) - (n + \delta + g)k = 0$$

$$\dot{k} = \frac{1}{2}\sqrt{400} - \frac{1}{20}400 = 10 - 20 = -10 \neq 0,$$

so the economy is not in its steady state.

3.2 Part B

When the immigration occurs, $L$ quadruples, and therefore $k = 100$. Then the change in $k$ is

$$\dot{k} = \frac{1}{2}\sqrt{100} - \frac{1}{20}100 = 5 - 5 = 0,$$

and so the economy is in a steady state.
4 Problem 4

Since both countries are in their steady states, we can solve for the steady state levels of capital in each country. Since $\delta$ and $n$ are the same across countries, we do not specify them with a subscript.

\[
\dot{k}_a = s_a A_a k_a^\alpha - (n + \delta) k_a = 0 \Rightarrow k_a^{ss} = \left( \frac{s_a A_a}{n + \delta} \right)^{\frac{1}{1-\alpha}}
\]

\[
\dot{k}_b = s_b A_b k_b^\alpha - (n + \delta) k_b = 0 \Rightarrow k_b^{ss} = \left( \frac{s_b A_b}{n + \delta} \right)^{\frac{1}{1-\alpha}}
\]

Since $y_a = y_b$, we have

\[
A_a \left( \frac{s_a A_a}{n + \delta} \right)^{\frac{\alpha}{1-\alpha}} = A_b \left( \frac{s_b A_b}{n + \delta} \right)^{\frac{\alpha}{1-\alpha}}
\]

\[
A_a^{\frac{1-\alpha}{\alpha}} s_a A_a = A_b^{\frac{1-\alpha}{\alpha}} s_b A_b
\]

\[
A_a^{\frac{1}{\alpha}} s_a = A_b^{\frac{1}{\alpha}} s_b \Rightarrow \frac{s_a}{s_b} = \left( \frac{A_b}{A_a} \right)^{\frac{1}{\alpha}} = 2^{\frac{1}{\alpha}}
\]

5 Problem 5

5.1 Part A

People consume a fraction $c$ of their income $C = cf(k)$, so saving (and therefore gross investment) is $S = f(k) - cf(k) - p = sf(k) - p$. The new differential equation governing the evolution of the per capita stock of capital in this country is then:

\[
\dot{k} = sf(k) - p - (n + \delta)k
\]

5.2 Part B

There will be two equilibria. One will occur at $k_l > 0$ and will be unstable, while the other will occur at $k_h > k_l > 0$. We get this because the graph of $(n + \delta)k$ now shifts up by $p$ (or the graph of $sf(k)$ shifts down by $p$, depending on what you use as your scale on the $y$ axis).

6 Problem 6

General capital evolution for no technological change is:

\[
\dot{k} = sk^\alpha - (n + \delta)k
\]

The steady state for this equation is

\[
k^{ss} = \left( \frac{s}{n + \delta} \right)^{\frac{1}{1-\alpha}}
\]

Therefore, for country $A$ we have

\[
k_{a}^{ss} = \left( \frac{.2}{.02} \right)^{\frac{1}{1-\alpha}} = 10^2 = 100 \Rightarrow y_{a}^{ss} = 10,
\]
and for country $B$ we have

$$k_{ss}^a = \left( \frac{A}{.02 + .02} \right)^{\frac{1}{1-a}} = 10^2 = 100 \Rightarrow y_{ss}^b = 10.$$  

In order to draw the graph of income per capita over time, we need the initial $y_0$ for each country, $y_{ss}$ for each country, and their relative growth rates. We have seen that both countries started at the same level and will eventually end up at the same steady state. It only remains to find their relative growth rates at time 0.

We are given the intensive form of the production function and that $y_0 = 5$ for both countries.

$$y_0 = k_0^\frac{1}{2} \Rightarrow k_0 = y_0^2 = 5^2 = 25$$

This implies that for country $A$

$$\frac{\dot{k}_0^a}{k_0} = s (k_0^a)^{\alpha - 1} - (n + \delta) = .2 (25)^{\frac{1}{2}} - (.02) = .02 \Rightarrow \frac{\dot{y}_0}{y_0} = .01$$

For country $B$ we have

$$\frac{\dot{k}_0^b}{k_0} = s (k_0^b)^{\alpha - 1} - (n + \delta) = .4 (25)^{\frac{1}{2}} - (.04) = .04 \Rightarrow \frac{\dot{y}_0}{y_0} = .02$$

Therefore, country $B$ is growing faster initially, and will converge to its steady state faster than country $A$.

### 7 Problem 7

Let’s consider the base situation:

$$\dot{k} = .6y - (n + \delta)k = .6k^5 - (n + \delta)k \tag{2}$$

At steady state the rate of change is zero, so:

$$0 = .6k^5 - (n + \delta)k$$

$$ .6k^5 = (n + \delta)k $$

$$ k^* = (\frac{.6}{(n + \delta)})^2 = \frac{.36}{(n + \delta)^2} $$

So output steady state is conveniently:

$$y^* = \frac{.6}{(n + \delta)}$$

The amount of consumption done is simply $(1 - s)$ of the output, or:

$$c^* = .4 \frac{.6}{(n + \delta)} = \frac{.24}{(n + \delta)}$$

In general, we can write the output steady state as:
\[ y^* = \frac{s}{n + \delta} \]

And the consumption steady state as:
\[ c^* = (1 - s) \frac{s}{n + \delta} = \frac{(1 - s)s}{(n + \delta)} \]

And the capital steady state as:
\[ k^* = \left( \frac{s}{n + \delta} \right)^2 = \frac{s^2}{(n + \delta)^2} \]

So now let’s consider out two alternative choices of savings rate.

**Plan A** - Lower savings to .5

In this case, we know our final steady state values of output, consumption and capital:

\[ y^*_A = \frac{.5}{n + \delta} \]
\[ c^*_A = \frac{.25}{n + \delta} \]
\[ k^*_A = \frac{.25}{(n + \delta)^2} \]

So we’d have greater steady state consumption under Plan A than with no change. How does consumption respond immediately and over time, though, to our change in policy? Notice that the steady state value of capital necessary to sustain Plan A is lower than the steady state value before the plan. So we need less capital. So we can raise consumption immediately and permanently by lowering our saving rate, letting some capital depreciate without replacement, and lowering our capital stock. Consumption will then asymptote down to it’s steady state level.

Note that immediately after the change, we have a very high consumption. Capital is still at it’s previous steady state value, so:

\[ k = \frac{.36}{(n + \delta)^2} \]
\[ y = \frac{.6}{n + \delta} \]

So with the lower saving rate:
\[ c = (1 - .5) \frac{.6}{n + \delta} = \frac{.30}{n + \delta} > c^*_A \]

**Plan B** - Lower savings to .4.

Similar analysis shows that we get the following steady states:

\[ y^*_B = \frac{.4}{n + \delta} \]
\[ c^*_B = \frac{.24}{n + \delta} \]
\[ k_B^* = \frac{.16}{(n + \delta)^2} \]

First, note that consumption in steady state is equal to our situation before the change. But capital is much lower in steady state. So we’d have an immediate large jump in consumption (larger than under plan A), followed by a falling level of consumption until we were right back where we started.

Note again the immediate effect on consumption

\[ k = \frac{.36}{(n + \delta)^2} \]

\[ y = \frac{.6}{n + \delta} \]

So with the lower saving rate:

\[ c = (1 - .4) \frac{.6}{(n + \delta)} = \frac{.36}{(n + \delta)} > c_B^* \]

8 Problem 8

First, a few quick notes on growth rates of different variables in the Solow model. In steady state, we’ll always have that:

\[ \frac{\dot{Y}}{Y_L} = \frac{\dot{K}}{K_L} = 0 \]

\[ \frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = g \]

\[ \dot{Y} = \frac{\dot{K}}{K} = n + g \]

Basically, what does an increase in \( g \) do? It causes capital per effective unit of labor to depreciate faster. (For brevity, when I say depreciation, I mean all three factors: population, depreciation, and technological change). That is, the depreciation line tilts upwards. Since we began at a steady state, depreciation was exactly equal to savings. Now, depreciation has immediately risen, and so our savings are insufficient to cover depreciation. So, immediately capital per effective unit starts decreasing. Because output is a monotonic function of capital, we have that output per effective unit of labor must start decreasing as well.

So, output per effective unit of labor starts decreasing. What next? With higher depreciation, the steady state level of capital per effective unit of labor will be lower. So output per effective unit of labor will be lower. So output per effective unit of labor will gradually fall until it hits the new steady state level.

Now, what happens to output per capita? This was originally growing at a constant rate \( g \). Now \( g \) has risen to \( g' \). We know that in the new steady state, output per capita will grow at \( g' \). What happens immediately and then in transition to the new steady state? Well, immediately, output per effective unit of labor is decreasing. Consider that we can write the growth rate of output per effective unit as follows:

\[ \frac{\dot{Y}}{eL} = \frac{\dot{Y}}{Y} \frac{Y}{Y_L} - \frac{\dot{e}}{e} \frac{Y}{Y_L} = \frac{\dot{Y}}{Y} \frac{Y}{Y_L} - g' \]
If output per efficiency unit is decreasing, then the growth rate must be negative. So we must have that:
\[
\left(\frac{\dot{Y}}{Y}\right) < g'
\]

Now, also consider that immediately following the change, the growth rate of output per capita must have jumped up. Why? If not, then it would have to be that output was still growing at only \(g + n\), as capital is unchanged immediately. But this does not allow for the change in technology growth. So we must have that immediately:
\[
g < \left(\frac{\dot{Y}}{Y}\right) < g'
\]

Why doesn’t output per capita jump immediately to \(g'\)? Because the immediate dilution of capital per effective unit has kept output from growing quite as quickly as technology.

Can we say anything about the transition to the steady state? Not really. Not even with fancy math is it possible to pin down the exact behavior of output.

**Addition to Answer Key**

There was some question in section as to the actual effect on the growth rate of income per capita. There was speculation that it would actually drop below \(g\) first, and then rise over time to \(g'\). It turns out that we can show mathematically that this is not the case. Growth in output per capita will jump immediately to some value in between \(g\) and \(g'\).

Consider that we know exactly the level of capital per efficiency unit at the original steady state. It is:
\[
\frac{K}{eL_{ss}} \equiv k_{ss} = \left(\frac{s}{n + \delta + g}\right)^{\frac{1}{1-\alpha}}
\]

Now the growth rate of technology changes to \(g'\). What will be the instantaneous effect on growth in output per capita? Let’s look first at the growth rate of output per effective unit. To do that we’ll look at the growth rate of capital per effective unit.

Immediately after the change, the equation governing the evolution of capital is:
\[
\dot{k} = s k^\alpha - (n + \delta + g') k
\]

And the growth rate is found by dividing both sides by \(k\).
\[
\frac{\dot{k}}{k} = s k^{\alpha - 1} - (n + \delta + g')
\]

Immediately after the change to \(g'\), capital per effective unit is still \(k_{ss}\), so plug that into this equation for capital’s growth rate:
\[
\frac{\dot{k}}{k} = s \left[\frac{s}{n + \delta + g}\right]^{\frac{1}{1-\alpha}} - (n + \delta + g')
\]
\[
= s \left(\frac{s}{n + \delta + g}\right)^{\frac{\alpha - 1}{1-\alpha}} - (n + \delta + g')
\]
\[
= s \left(\frac{s}{n + \delta + g}\right)^{-1} - (n + \delta + g')
\]
\[ n + \delta + g - (n + \delta + g') = g - g' < 0 \]

Which is a pretty simple representation of the fact that capital per effective unit is shrinking.

Now, we need to evaluate the growth rate of output per effective unit. We can arrive at this from the intensive form of the production function:

\[ y = k^\alpha \]

Where \( y \) and \( k \) are both in per effective unit terms. Taking logs and then differentiating with respect to time, we can show that:

\[ \frac{\dot{y}}{y} = \frac{\dot{k}}{k} \]

So we can easily see that after the change to new technology, we’ll have the growth rate of output per effective unit as:

\[ \frac{\dot{y}}{y} = \frac{\dot{k}}{k} = \alpha(g - g') < 0 \]

So output per effective unit is shrinking as well, but at a slower rate than capital per effective unit. Now we can use this fact about the growth rate of output per effective unit to understand the growth rate of output per capita. Notice that:

\[ \frac{\dot{y}}{y} = \frac{\dot{\tilde{y}}}{\tilde{y}} = \frac{\dot{\tilde{y}}}{\tilde{y}} - \frac{\dot{\tilde{e}}}{\tilde{e}} \]

Which can be rearranged to show that growth rate in output per capita is:

\[ \frac{\dot{\tilde{y}}}{\tilde{y}} = \frac{\dot{y}}{y} + \frac{\dot{\tilde{e}}}{\tilde{e}} \]

Plugging in values for growth rate of output per effective unit from above, and for the new growth rate of technology, we have:

\[ \frac{\dot{\tilde{y}}}{\tilde{y}} = \alpha(g - g') + g' = \alpha g + (1 - \alpha)g' \]

Or growth in output per capita is simply a weighted average of the two growth rates. But what we can say for certain because we know that \( g < g' \), is that:

\[ g < \frac{\dot{\tilde{y}}}{\tilde{y}} < g' \]

So the growth rate of output per capita will jump upwards immediately to some level in between \( g \) and \( g' \), and will then rise over time until it asymptotes to \( g' \).
9 Problem 9

9.1 Part A

Since $n$, $\delta$, $s$, and $g$ are the same for all countries, we have that

$$k_m = \left(\frac{sA_m}{n+\delta+g}\right)^{\frac{1}{1-\alpha}} \Rightarrow y_m = A_m \left(\frac{sA_m}{n+\delta+g}\right)^{\alpha}$$

$$k_n = \left(\frac{sA_n}{n+\delta+g}\right)^{\frac{1}{1-\alpha}} \Rightarrow y_n = A_n \left(\frac{sA_m}{n+\delta+g}\right)^{\alpha}$$

We also have that

$$y_m = 5y_n$$

$$A_m \left(\frac{sA_m}{n+\delta+g}\right)^{\frac{1}{1-\alpha}} = 5A_n \left(\frac{sA_n}{n+\delta+g}\right)^{\frac{1}{1-\alpha}}$$

$$A_m^{\frac{1}{1-\alpha}} \left(\frac{s}{n+\delta+g}\right)^{\frac{1}{1-\alpha}} = 5A_n^{\frac{1}{1-\alpha}} \left(\frac{s}{n+\delta+g}\right)^{\frac{1}{1-\alpha}}$$

$$A_m^{\frac{1}{1-\alpha}} = 5A_n^{\frac{1}{1-\alpha}} \Rightarrow A_m = 5^{1-\alpha}A_n.$$  

Plugging $A_m$ into the equation for $k_m$ we have

$$k_m = \left(\frac{5^{1-\alpha}sA_n}{n+\delta+g}\right)^{\frac{1}{1-\alpha}} = 5k_n$$

9.2 Part B

As we saw in part A,

$$A_m = 5^{1-\alpha}A_n.$$  

9.3 Part C

$$f'(k_m) = \alpha A_m k_m^{\alpha-1} = \alpha 5^{1-\alpha} A_n (5k_n)^{\alpha-1} = \alpha A_n k_n^{\alpha-1} = f'(k_n)$$

10 Problem 10

The production function is given by

$$Y = A(bK)^\alpha (cL)^{1-\alpha}$$

$$0 < \alpha < 1$$

One might be inclined to form the model as usual, dividing everything by $cL$. This would imply

$$y' = A(bk)^\alpha$$

where

$$y' = \frac{Y}{cL}$$
This would imply that
\[ \gamma_{k'} = sA(k'^{\alpha-1}) - (n + g + \delta) \]  
(3)

There can be no steady state such that \( \gamma_{k'} = 0 \). If there were, that would imply that there were a fixed \( k^* \). However over time \( A \) and \( b \) would increase, making it such that 3 became positive. One can also show that there is no steady state such that \( \gamma_{k'} \) is constant and the growth of output per capita is constant.

As a result, we would like to reform the model so that all technology is labor augmenting.

\[ Y = A(bK)^\alpha(cL)^{1-\alpha} \]
\[ = A^{(1-\alpha)/(1-\alpha)}b^{\alpha(1-\alpha)/(1-\alpha)}K^{\alpha}(cL)^{1-\alpha} \]
\[ = K^{\alpha}(A^{1/(1-\alpha)}b^{\alpha/(1-\alpha)}cL)^{1-\alpha} \]

where
\[ e = A^{1/(1-\alpha)}b^{\alpha/(1-\alpha)}c \]

This allows us to form the standard intensive-form production function and work through the model in the usual fashion. Letting \( \ddot{y} \) denote output per worker, in steady state, we know that

\[ \gamma_{\ddot{y}} = \frac{\dot{e}}{e} \]
\[ = \frac{1}{1-\alpha} \frac{\dot{A}}{A} + \frac{\alpha}{1-\alpha} \frac{\dot{b}}{b} + \frac{\dot{c}}{c}. \]

Thus,
\[ \gamma_{\ddot{y}} = \frac{1}{1-\alpha} \gamma_A + \frac{\alpha}{1-\alpha} \gamma_b + \gamma_c \]

11 Problem 11

11.1 Part A

The intensive form model is given by

\[ y = k^\alpha \]
\[ \dot{k} = sk^\alpha - [n(k) + \delta]k \]

where
\[ n(k) = n_h \text{ if } k < (\ddot{y})^{1/\alpha} \]

and
\[ n(k) = n_l \text{ if } k > (\ddot{y})^{1/\alpha}. \]

There are three possible configurations of this model.
1. Investment per worker cannot overcome the effective depreciation for any level above \( k_1^* \), even above \( \bar{k} \) where this effective depreciation is relatively lower. In this case, if \( k_0 > k_1^* \) effective depreciation will always draw the economy down to \( k_1^* \). If \( k_0 < k_1^* \) saving is enough to increase capital per worker up to \( k_1^* \). Thus, \( k_1^* \) is the unique globally stable steady state.

2. In this case, saving is sufficiently high such that the economy can climb out of the high population growth regime for any \( k \) below \( \bar{k} \). As is capital per worker passes above \( \bar{k} \), population, and so effective depreciation, declines, giving the economy an extra boost as it climbs to \( k_2^* \). As usual, the marginal product of capital is declining in \( k \), so eventually increases in \( k \) lead to smaller increases in investment. So for \( k > k_2^* \) effective depreciation overcomes investment, and capital per worker declines to \( k_2^* \). As a result, \( k_2^* \) is the unique globally stable steady state.

3. The saving rate is such that the economy could sustain a steady state in the low population growth regime, denoted by \( k_{3,b}^* \). However, the saving rate is not high enough such that investment per capita unconditionally overcomes effective depreciation in the high population growth regime. Thus for \( k_0 < \bar{k} \), the economy is drawn to the high population growth steady state, \( k_{3,a}^* \), which is locally stable. And for \( k_0 > \bar{k} \), the economy stays in the high population growth regime, heading to \( k_{3,b}^* \).

This situation with multiple equilibria is an example of a poverty trap: The economy cannot overcome some threshold level of output per worker, above which a higher level of output per worker would be enjoyed. Additionally, it is an example of conditional club convergence. Club convergence is defined a situation in which economies that only differ in their levels of initial capital may head to different steady states.

### 11.2 Part B

For there to be a unique equilibrium under the high population growth regime, we need that the economy cannot sustain any level of capital per worker under the low population growth regime. It is necessary and sufficient that we require even the threshold level of capital is unaffordable to maintain.

\[
(n_l + \delta) \bar{k} > s \bar{k}^\alpha
\]

This implies that

\[
s < (n_l + \delta) \bar{k}^{1-\alpha}
\]

\[
s < (n_l + \delta) \bar{y}^{(1-\alpha)/\alpha}
\]

For a unique equilibrium under the low population growth regime, we need the economy to oversave at every point where it still has high population growth. Again, a necessary and sufficient condition can be obtained by looking at the threshold capital per worker level. At \( \bar{k} \) the economy must just invest enough that it can sustaining \( \bar{k} \) and so jump to the low population growth regime.

\[
s \bar{k}^\alpha \geq (n_h + \delta) \bar{k}
\]

As a result,

\[
s \geq (n_h + \delta) \bar{k}^{1-\alpha}
\]

\[
s \geq (n_h + \delta) \bar{y}^{(1-\alpha)/\alpha}
\]

For multiple equilibria, we require that under high population growth the economy cannot make it to \( \bar{k} \) and so switch regimes. Further we need some level of capital to be affordable under the
low population growth set up. It again suffices to check the affordability of $\bar{k}$. Thus, for multiple equilibria we need

$$(nt + \delta)\bar{y}^{(1-\alpha)/\alpha} < s < (nh + \delta)\bar{y}^{(1-\alpha)/\alpha}$$

12 Problem 12

12.1 Part A

For country 1, post tax income is $f(k)(1-\tau)$, so the differential equation governing the evolution of the capital stock is

$$\dot{k} = s [f(k)(1-\tau)] - (n + \delta)k.$$ 

For country 2, post tax income is $f(k) - \psi$, so the differential equation governing the evolution of the capital stock is

$$\dot{k} = s [f(k) - \psi] - (n + \delta)k.$$ 

The per-household tax in country 1 is $\tau f(k)$, and the per household tax in country 2 is $\psi$, so we have that

$$\psi = \tau f(k)$$

12.2 Part B

In country 1, we can solve for the steady state level of capital

$$(1-\tau)sk^\alpha = (n+\delta)k$$

$$k_{ss} = \left(\frac{(1-\tau)s}{n+\delta}\right)\frac{1}{1-\alpha} \Rightarrow y_{ss} = \left(\frac{(1-\tau)s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

This implies that the tax rate is

$$\psi = \tau \left(\frac{(1-\tau)s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Re writing the differential equation governing the evolution of the capital stock

$$\dot{k} = sf(k) - (n + \delta)k - s\tau \left(\frac{(1-\tau)s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Notice that the graph of country one starts at the origin and has the usual shape, although with a lower savings rate than the second country. The second country, however, is shifted down, making two non-zero steady states. These can intersect in many interesting ways, depending on the parameters of the problem.

13 Problem 13

The problem gives us the differential equations governing the evolution of capital and human capital in the model:

$$\dot{k} = sf(k,h) - (\delta + n)k = sk_k^\frac{1}{2} h^\frac{1}{2} - (\delta + n)k$$

$$\dot{h} = sf(k,h) - (\delta + n)h = sh_h^\frac{1}{2} h^\frac{1}{2} - (\delta + n)h$$
In the steady state we have
\[ s_k k^{\frac{1}{3}} h^{\frac{1}{3}} = (\delta + n)k \]
\[ s_h k^{\frac{1}{3}} h^{\frac{1}{3}} = (\delta + n)h \]
Dividing one equation by the other, we have that
\[ \frac{s_k}{s_h} = \frac{k}{h} \]
We use this to plug in for the steady state level of \( k \) in the differential equation for \( h \)
\[ s_h \left( \frac{s_k}{s_h} \right)^{\frac{2}{3}} h^{\frac{2}{3}} = (\delta + n)h \Rightarrow s_h \left( \frac{s_k}{s_h} \right)^{\frac{2}{3}} = (\delta + n)h^{\frac{1}{3}} \]
\[ h_{ss} = \left( \frac{s_h}{\delta + n} \right)^{\frac{3}{2}} \frac{s_k}{s_h} = \frac{s_h^2 s_k}{(\delta + n)^3} \]
Plugging this into the equation for \( \dot{k} \)
\[ s_k k^{\frac{2}{3}} \left( \frac{s_h^2 s_k}{(\delta + n)^3} \right)^{\frac{1}{3}} = (\delta + n)k \Rightarrow s_k \left( \frac{s_h^2 s_k}{(\delta + n)^3} \right)^{\frac{1}{3}} = (\delta + n)k^{\frac{2}{3}} \]
\[ k_{ss} = \left( \frac{s_k}{\delta + n} \right)^{\frac{2}{3}} \left( \frac{s_h^2 s_k}{(\delta + n)^3} \right)^{\frac{1}{3}} = \frac{s_h s_k^2}{(\delta + n)^2} \]
Plugging these two expressions into the production function, we get the steady state level of output:
\[ y = k^{\frac{1}{2}} h^{\frac{1}{2}} = \left( \frac{s_h s_k^2}{(\delta + n)^3} \right)^{\frac{1}{3}} \left( \frac{s_h^2 s_k}{(\delta + n)^3} \right)^{\frac{1}{3}} = \frac{s_h s_k}{(\delta + n)^2} \]
Therefore, if two countries have the same savings rate for capital, but one has twice the savings rate for human capital, then it will have twice the steady state output.

### 14 Problem 14

#### 14.1 Part A

First let’s look at the steady state \( (\dot{k} = 0) \) for country 1:
\[ 0.1 k^{0.5} - 0.1 k = 0 \quad \Leftrightarrow \quad k_{SS}^1 = 1 \]
and for country 2:
\[ 0.2 \frac{1}{1 + k} k^{0.5} - 0.1 k = 0 \quad \Leftrightarrow \quad \frac{1}{1 + k} k^{-0.5} = \frac{1}{2} \]
This is a bit hard to solve. If we check that the solution for country 1, we see that it is also a solution for country 2 so: \( k_{SS}^2 = 1 \).

Intuitively, country 2 moves faster to the steady state. To see this, notice that in their respective steady states, both countries have a savings rate of \( s = 0.1 \). This will be constant for all points in time for country 1, but if country 2 has more capital than its steady state, its savings rate will be lower, and therefore \( k \) will be falling faster towards its steady state. If country 2 has less capital than its steady state, its savings rate will be higher, and therefore \( k \) will be rising faster towards its steady state. The part B shows this result mathematically.
14.2 Part B

We know \( y = k^\alpha \) so taking logs we know \( \ln y = \alpha \ln k \) in particular in the steady state \( \ln y_{SS} = \alpha \ln k_{SS} \), so:

\[
\ln y - \ln y_{SS} = \alpha (\ln k - \ln k_{SS}) \quad \Leftrightarrow \quad \frac{1}{\alpha} (\ln y - \ln y_{SS}) = \ln k - \ln k_{SS} \tag{4}
\]

Differentiating \( \ln y = \alpha \ln k \) with respect to time we get \( \frac{\dot{y}}{y} = \frac{k}{k} \). In the standard Solow model the growth rate of capital is given by:

\[
\frac{\dot{k}}{k} = s k^{\alpha - 1} - (n + \delta) \tag{5}
\]

So:

\[
\frac{\dot{y}}{y} = \alpha [s k^{\alpha - 1} - (n + \delta)] = \alpha [s e^{(\alpha - 1) \ln k} - (n + \delta)]
\]

Which we can use a first order Taylor series expansion:

\[
\frac{\dot{y}}{y} = \alpha [s e^{(\alpha - 1) \ln k^*} - (n + \delta)] + \alpha [(\alpha - 1) s e^{(\alpha - 1) \ln k^*}] \ln k - \ln k^*]
\]

Notice that in steady state (from equation 5) is \( sk^{* (\alpha - 1)} = (n + \delta) = se^{(\alpha - 1) \ln k^*} \). So (and using equation 4):

\[
\frac{\dot{y}}{y} = \alpha (\alpha - 1)(n + \delta)[\ln k - \ln k^*] = (\alpha - 1)(n + \delta)[\ln y - \ln y^*] \tag{6}
\]

Similarly the growth rate of income in country 2 is:

\[
\frac{\dot{y}}{y} = \alpha [s_0 \frac{1}{1 + k} k^{\alpha - 1} - (n + \delta)] = \alpha [s_0 \frac{1}{1 + e^{\ln k}} e^{(\alpha - 1) \ln k} - (n + \delta)]
\]

\[
\frac{\dot{y}}{y} = \alpha [s_0 \frac{1}{1 + e^{\ln k}} e^{(\alpha - 1) \ln k^*} - (n + \delta)] + \alpha [s_0 \frac{-e^{\ln k^*}}{(1 + e^{\ln k^*})^2} e^{(\alpha - 1) \ln k^*}]
\]

\[
\frac{\dot{y}}{y} = \left[ -s_0 \frac{-e^{\ln k^*}}{(1 + e^{\ln k^*})^2} + (\alpha - 1)(n + \delta)][\ln y - \ln y^*] \right] \tag{7}
\]

Comparing 6 with 7 we can see that \( \gamma_2 > \gamma_1 \).

15 Problem 15

The log-linearized growth rate of output per unit of effective labor, as derived in the lecture notes, is

\[
\hat{\gamma}_y = -\gamma \ln \left[ \frac{y_0}{y^*} \right] \tag{8}
\]

where
\[ \gamma = (1 - \alpha)(n + g + \delta) \]  

If \( y \) is at some fraction, \( 1/c \), of its steady state level

\[ \hat{\gamma}_y(y^*/c) = \gamma \ln c \]

This implies that output per capita, \( \bar{y} \), should be growing at the rate:

\[ \hat{\gamma}_\bar{y}(y^*/c) = \gamma \ln c + g \]

Given the parameters of the model,

\[ \hat{\gamma}_\bar{y}(y^*/4) \approx 0.0847 \]  

and

\[ \hat{\gamma}_\bar{y}(y^*/2) \approx 0.0523 \]  

To determine the time to get from \( y^*/4 \) to \( y^*/2 \), we use the time path of \( y \) based on log-linearization:

\[ \ln(y_t) = e^{-\gamma t} \ln(y_0) + (1 - e^{-\gamma t}) \ln(y^*) \]

so

\[ \ln(y_t) - \ln(y^*) = e^{-\gamma t}[\ln(y_0) - \ln(y^*)] \]

\[ \ln(\frac{y_t}{y^*}) = e^{-\gamma t} \ln(\frac{y_0}{y^*}) \]  

Using \( y_t = y^*/2 \) and \( y_0 = y^*/4 \),

\[ \ln(1/2) = e^{-\gamma t} \ln(1/4) \]

\[ -\gamma t = \ln[\frac{\ln(1/2)}{\ln(1/4)}] \]

\[ t = -\frac{1}{\gamma} \ln[\frac{\ln 2}{\ln 4}] \]

\[ = \frac{1}{\gamma} \ln 2 \]

\[ \approx 14.85 \]

To test the log-linear approximation of the growth rate compute the actual growth rate of \( y \) at some fraction of its steady state level.

\[ \gamma_y(y^*/c) = \alpha \gamma_k(y^*/c) \]

\[ = \alpha[s(k|_{y=y^*/c})^{-(1-\alpha)} - (n + g + \delta)] \]

\[ = \alpha[s(\frac{y^*}{c})^{-(1-\alpha)/\alpha} - (n + g + \delta)] \]
The steady state level of $y$ is given by

$$y^* = \left[ \frac{s}{n + g + \delta} \right]^{\alpha/(1-\alpha)}$$

So we may re-write

$$\gamma_y(y^*/c) = \alpha[sc^{(1-\alpha)/\alpha}(y^*)^{-(1-\alpha)/\alpha} - (n + g + \delta)]$$

$$= \alpha[sc^{(1-\alpha)/\alpha} \frac{n + g + \delta}{s} - (n + g + \delta)]$$

$$= \alpha(n + g + \delta) \left[ c^{(1-\alpha)/\alpha} - 1 \right]$$

$$= \frac{0.07}{3} [c^2 - 1]$$

As a result, the growth rate of output per capita is

$$\gamma_y(y^*/c) = \frac{0.07}{3} [c^2 - 1] + g$$

Using this the actual growth rates of output per capita at $1/4$ and $1/2$ of the steady state are

$$\gamma_y(y^*/4) = 0.37 > 0.0847 \approx \hat{\gamma}_y(y^*/4)$$

and

$$\gamma_y(y^*/2) = 0.09 > 0.0523 \approx \hat{\gamma}_y(y^*/2)$$

Thus, we see that the log-linear approximation of the growth rate understates the true growth rate and is worse farther away from the steady state. This will hold true in general. As a result, our doubling time estimate is probably too high. To get a better estimate of this doubling time, we can simulate the capital motion equation in discrete time as

$$\frac{k_{t+1} - k_t}{k_t} = s(k_t)^{\alpha-1} - (n + g + \delta)$$

$$k_{t+1} = s(k_t)^\alpha + [1 - (n + g + \delta)]k_t$$

This implies that

$$y_{t+1} = (k_{t+1})^\alpha$$

$$= [s(k_t)^\alpha + [1 - (n + g + \delta)]k_t]^\alpha$$

$$= [sy_t + [1 - (n + g + \delta)](y_t)^{1/\alpha}]^\alpha$$

At $t = 0$, we start with $y_0 = y^*/4 \approx 0.423$. Using our discrete simulation, $y_0 = 0.899 > 0.845 \approx y^*/2$. Thus we conclude that the doubling time of 14 years we calculated based on log-approximation is pretty poor.
16 Problem 16

First note that because capital can move freely between countries marginal product of capital is the same in both countries

$$\text{MPK}_A = \text{MPK}_B = \alpha \bar{k}^{\alpha-1} \Rightarrow k_A = K_B = \bar{k}$$

and people in both countries can only hold capital as assets, so

$$2\bar{k} = \frac{a_A + a_B}{2} \text{ and } \text{MPK} = \alpha \left( \frac{a_A + a_B}{2} \right)^{\alpha-1} = r.$$ 

In a closed economy people can only hold capital in their own country, yet in an open economy people hold assets, which is capital in either their country or abroad. So similarly as in a closed economy:

$$\dot{a}_i = \text{Income} - \text{Consumption} - (n + \delta + g)a_i$$

That is people accumulate assets by saving (= income - consumption) minus depreciation of assets, population growth and technological progress. In an open economy

$$\text{Income} = wL + ra_i,$$

that is the sum of wages of all people in the economy, plus the real rate of interest times the amount of assets they hold. From the problem we are given that

$$\text{Consumption} = wL + (1 - s_i)ra_i \Rightarrow$$

$$\dot{a}_i = s_i ra_i - (n + \delta + g)a_i$$

Now we can solve for the steady state

$$s_i \alpha \left( \frac{a_A + a_B}{2} \right)^{\alpha-1} a_i - (n + \delta + g)a_i = 0$$

$$s_i \alpha \left( \frac{a_A + a_B}{2} \right)^{\alpha-1} - (n + \delta + g) = 0 \quad \lor \quad a_i = 0$$

If you try to solve for both countries you will see that the only possible solution is (note that in our problem $\delta = g = 0$) $a_A = 0$ and $a_B = 2 \left( \frac{\alpha s_B}{n} \right)^{\frac{1}{1-\alpha}}$.

To see this, notice that from the law of motion $\dot{a}_i$, in the steady state we have

$$s_A \alpha \left( \frac{a_A + a_B}{2} \right)^{\alpha-1} = n$$

$$s_B \alpha \left( \frac{a_A + a_B}{2} \right)^{\alpha-1} = n$$

Which implies $s_A = s_B$, a contradiction. Therefore, in the “steady state”, at least one country will not have $\dot{a} = 0$!

How is this possible? Remember back to the open economy interest rate problems in the consumption portion of the course. We had that if the discount rate was different between two countries, the more patient country would eventually soak up all the consumption in the world, while the more impatient country would eventually approach zero consumption. Something similar happens here.

Just as in the other problem, we can eliminate certain cases by contradiction.
1. If either $\dot{a}_A$ or $\dot{a}_B$ are greater than zero, then we will have ever increasing assets and capital. However, due to population growth, we know that there will be only a finite level of capital in the world, therefore we reach a contradiction.

2. If both are less than zero, then we end up with no capital, which is also a contradiction.

3. We have already eliminated the case where both are zero.

The only case left is where one is negative, and the other goes to zero. The country whose asset growth is negative will converge to $a = 0$, and the other country will have all the assets in the world, which will converge to some steady level $a = \bar{a}$, since we have that in the steady state its assets grow at rate 0.

We now show that country A’s assets go to zero, not country B’s. Notice that from the law of motion for assets,

$$\frac{\dot{a}_B}{a_B} = s_B r - n > s_A r - n = \frac{\dot{a}_A}{a_A}.$$ 

Since one country has 0 asset growth and the other has negative assets growth, and country B’s assets grow faster than country A’s, country A has the negative rate of asset growth, and therefore will have no assets in the long run.

We solve for country B’s steady state level of assets by plugging in $\dot{a}_B = 0$ and $a_A = 0$ to get

$$s_B \alpha \left( \frac{a_B}{2} \right)^{\alpha - 1} = n$$

$$a_B = \left( \frac{\alpha s_B}{n} \right)^{\frac{1}{1-\alpha}}$$

17 Problem 17

We know the two primary equations of the Ramsey model

$$\frac{\dot{c}}{c} = \frac{1}{\sigma} \left( f'(k) - n - \delta - \theta \right)$$

$$\dot{k} = f(k) - c - (n + \delta)k$$

In steady state, both rates of change will be zero, so

$$\frac{1}{\sigma} (f'(k) - n - \delta - \theta) = 0$$

$$\frac{1}{2} k^{-\frac{1}{2}} = n + \delta + \theta$$

$$k^{-\frac{1}{2}} = 2(n + \delta + \theta)$$

$$k_{ss} = \frac{1}{4(n + \delta + \theta)^2}$$

For the $\dot{k}$ equation:

$$f(k) - c - (n + \delta)k = 0$$

$$c = f(k) - (n + \delta)k$$
\[ k^{\frac{1}{2}} - (n + \delta)k \]

Plug in \( k_{ss} \) from above:

\[ c = \frac{1}{2(n + \delta + \theta)} \left( n + \delta \right) - \frac{n + \delta}{4(n + \delta + \theta)^2} \]

\[ = \frac{2n + 2\theta + 2\delta - n - \delta}{4(n + \delta + \theta)} \]

\[ c_{ss} = \frac{n + 2\theta + \delta}{4(n + \delta + \theta)^2} \]

18 Problem 17.5

Notice that the agent wants to be in the steady state if time goes to infinity because it is the only sustainable level of consumption which gives the agent a non-infinite disutility. However, if time is finite, then the agent will want to consume all his assets at time \( T \), otherwise he could increase lifetime utility. Therefore, he will want to be at the \( k = 0 \) axis exactly at time \( T \). This means the agent will consume slightly more than the stable arm at time \( t = 0 \), exactly enough to be at \( k = 0 \) at time \( T \).

19 Problem 18

If \( \theta_A > \theta_B \), then this implies than the \( \dot{c} = 0 \) axis for country A will be to the left of the \( \dot{c} = 0 \) axis for country B. Also, if they start of with the same level of capital, it must be the case that the initial consumption for country A will be greater than that of country B (Why? Because country A is more impatient, and wants to frontload consumption more than country B).

Given what we determined in the previous paragraph, it is completely feasible for the stable arm of country A to lie completely above stable arm for country B. However, can we say this with surety? Will the stable arms ever cross? We will show they do not using proof by contradiction.

Suppose they cross. Since the \( \dot{k} = 0 \) locus is the same for both countries, we will have that

\[ \dot{k}_A = k_A^{\alpha} - c_A - (n + \delta)k_A = k_B^{\alpha} - c_B - (n + \delta)k_B = \dot{k}_B \]

Why is this? Because at the point where the stable arms cross, both countries will be the same distance from the \( \dot{k} = 0 \) locus. Also, notice that \( k_A = k_B \) at this point and that that \( c_A = c_B \).

At this point, since the \( \dot{c} = 0 \) locus is different for the two countries, country B will be farther away from its \( \dot{c} = 0 \) locus than country A. Therefore

\[ \dot{c}_A < \dot{c}_B \Rightarrow \]

\[ \frac{1}{\sigma} \left( \alpha k_A^{\alpha - 1} - n - \delta - \theta_A \right) < \frac{1}{\sigma} \left( \alpha k_B^{\alpha - 1} - n - \delta - \theta_B \right) \Rightarrow \]

\[ \theta_A > \theta_B, \]

a contradiction. Therefore, the lines cannot cross.
20 Problem 18.5

20.1 Part A

We can rewrite our problem as:
\[
\max_{c_t, k_t} V = \int_0^\infty e^{-\theta t} U(c_t) (N_0 e^{nt})^\pi dt = N_0^\pi \int_0^\infty e^{(n\pi - \theta)t} U(c_t) dt
\]
s.t.
\[
\dot{k} = k^\alpha - c - nk
\]
For this to be solvable we need \((n\pi - \theta) < 0\).

20.2 Part B

Remember that the usual first order condition for the Ramsey model is
\[
\frac{\dot{c}}{c} = \frac{1}{\sigma} \left( f'(k) - n - \theta \right).
\]
In this case, however, the discount rate in the integrand is \(\theta - \pi n\), not just \(\theta\). Replacing the correct discount rate, and the fact that we have
\[
\frac{\dot{c}}{c} = \frac{1}{\sigma} \left( f'(k) - n - \theta + \pi n \right)
\]
In the steady state we have
\[
\frac{\dot{c}}{c} = \frac{1}{\sigma} \left( f'(k) - n - \theta + \pi n \right) = 0
\]
\[f'(k) = (1 - \pi)n + \theta \]
\[\alpha k^{\alpha-1} = (1 - \pi)n + \theta \]
\[k = \left( \frac{\alpha}{(1 - \pi)n + \theta} \right)^{\frac{1}{\alpha-1}}
\]

21 Problem 19

So, we have a Ramsey model where initial capital and initial consumption are identical in both countries. Therefore, the marginal product of capital \((r)\) must be the same in both. But, country one has a higher discount rate than country two. So country one will have a lower steady state level of consumption and a lower steady state level of capital. For both countries to be on their stable arms, the stable arms must cross. And if they cross, country one’s must be steeper. (Easiest to see on the diagram).

Knowing that the stable arm in country one is steeper, then we know that:
\[
\frac{\dot{c}_1}{c_1} > \frac{\dot{c}_2}{c_2}
\]
\[
\frac{1}{\sigma_1} (r - \theta_1) > \frac{1}{\sigma_2} (r - \theta_2)
\]
\[
\frac{(r - \theta_1)}{(r - \theta_2)} > \frac{\sigma_1}{\sigma_2}
\]
And knowing that $\theta_1 > \theta_2$, we know that

$$1 > \frac{(r - \theta_1)}{(r - \theta_2)}$$

And this directly implies that

$$1 > \frac{\sigma_1}{\sigma_2}$$

So country one must have a lower rate of intertemporal substitution than country two. The diagram of consumption versus time shows that country one trades high early consumption for lower overall steady state consumption. Country two takes lower consumption early in exchange for higher steady state consumption.

### 22 Problem 19.5

#### 22.1 Part A

Notice that because we have depreciation $r = f'(k_t) - \delta = A - \delta$. So it is straightforward that as in the usual Ramsey model:

$$\dot{c} = \frac{1}{\alpha}(A - \delta - \theta)$$

and

$$\dot{k} = Ak - c - \delta k$$

#### 22.2 Part B

Notice first that because we are given that $A - \delta > \theta$, we know that $\dot{c} > 0$ always (so there won’t be any $\dot{c} = 0$ locus and we will have consumption increasing everywhere). When it come to capital, notice that $\dot{k} = 0 = (A - \delta)k - c$ is a straight line ($c = (A - \delta)k$), with slope $A - \delta$, which is positive because we are given $A - \delta > \theta > 0$. So now the question is how does $k$ behave, away from the $\dot{k} = 0$ locus. Suppose that $c$ and $k$ are such that you are in the line $\dot{k} = 0$. Now suppose that fixed $k$ you increase $c$ so that above $\dot{k} = 0$ locus you have $\dot{k} < 0$. Similarly below $\dot{k} = 0$ you have $\dot{k} > 0$.

#### 22.3 Part C

If you look at the graph from the previous part, you should notice that there is the possibility of a balanced growth path below the $\dot{k} = 0$ locus (where capital and consumption are growing forever). In this case the balanced growth path is given by $\dot{k} = \dot{c}$, that is capital and consumption are growing by the same amount (not ratio as you have seen before). When $\dot{k} = \dot{c}$ we have a growth path parallel to the $\dot{k} = 0$ locus, given by $c = (A - \delta)k - \frac{1}{\alpha}(A - \delta - \theta)$ and this will be the path chosen by the social planner (it is the only path that can achieve growth forever).
22.4 Part D

Notice that because the production function is AK, we have \( \dot{y} = \dot{k} = \dot{c} = \gamma \). So in both countries output will be growing by a fixed amount (equal in both countries). So we have:

\[
\frac{y_1 + \gamma t}{y_2 + \gamma t} \bigg|_{t \to \infty} \to 1
\]

To see that this is true, suppose \( y_1 = 2 \), \( y_2 = 1 \) and they both grow at \( \gamma = 1 \). Then:

\[
t = 0 \implies \frac{y_1}{y_2} = 2
\]

\[
t = 1 \implies \frac{y_1}{y_2} = \frac{3}{2}
\]

\[
t = 2 \implies \frac{y_1}{y_2} = \frac{4}{3}
\]

\[
t = 3 \implies \frac{y_1}{y_2} = \frac{5}{4}
\]

And so on, converging to 1.

22.5 Part E

Because we were able to solve for the balanced growth path \( (c = (A - \delta)k - \frac{1}{\alpha}(A - \delta - \theta)) \), then given \( k(0), c(0) = (A - \delta)k(0) - \frac{1}{\alpha}(A - \delta - \theta) \).

23 Problem 20

It’s a Ramsey model so start with the basic Ramsey equations:

\[
\frac{\dot{c}}{c} = \frac{1}{\sigma}(f'(k) - n - \delta - \theta)
\]

\[
\dot{k} = f(k) - c - nk - G
\]

Notice that only the law of motion for the capital stock changes with a change in \( G \).

What happens to the \( \dot{k} = 0 \) equation when \( G \) increases?

\[
0 = f(k) - c - nk - G
\]

\[
c = f(k) - nk - G
\]

Therefore, for any level of \( k \), consumption is now lower with an increased \( G \).

Things to note:

- Capital cannot jump (unless it is seized or destroyed by natural disaster)
- Consumption can jump, but only when there is new information
Knowing this, let’s start with a series of simpler problems and build up to the problem where spending oscillates between $G_h$ and $G_l$.

1. Let’s assume that prior to some time $t_1$ the economy is in a steady state with $G_l$. Suddenly, at $t_1$ $G$ increases to $G_h$. In this case, consumption will jump to the stable arm of the new steady state, which happens to be the steady state itself for this example (since steady state capital does not change with a change in $G$ according to the first order condition).

2. Now assume we are in a steady state prior to $t_1$, and at $t_1$ it is announced that effective immediately $G$ increases to $G_h$, and effective $t_2$ $G$ will decrease back to $G_l$. Instead of consumption jumping down to the new steady state immediately, it will only fall partially since the social planner knows that the government spending will decrease again. So the social planner will only lower consumption such that the dynamics will pull the economy up and left so that at time $t_2$, the economy will be on the old stable path towards the old steady state, and converge towards it.

3. Now assume that it is announced at time $t_1$ that we will have $G_h$ at $t_1$, $G_l$ at $t_2$, and $G_h$ at $t_3$. Then we will lower consumption, but not all the way to the steady state, ride the dynamics so that at time $t_2$ we are NOT on the stable arm. When we switch at time $t_2$, the dynamics will now pull us up and left until we cross the $\dot{c} = 0$ locus, and then will pull us down and left. We plan it such that at time $t_3$, we are on the stable arm towards the steady state with $G_h$.

4. Now assume they oscillate continually. Then we will go round and round in circles between the two steady states. Short cycles imply a small circle, long cycles big circles.

What happens to interest rates in all these examples? Since $r = f'(k)$, when $k$ increases, then $r$ falls. Follow the dynamics and you can graph $r$.

**24  Problem 21**

First the consumption equation:

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}(f'(k) - n - \theta) = \frac{1}{\sigma}(A\alpha k^{\alpha-1} - n - \theta) \quad (16)$$

We solve for $\dot{c} = 0$ locus:

$$A\alpha k^{\alpha-1} - n - \theta = 0$$

$$k_{ss} = \left(\frac{A\alpha}{n + \theta}\right)^{\frac{1}{\alpha}}$$

Therefore, a decrease in $A$ will cause a decrease in $k_{ss}$, so the $\dot{c} = 0$ shifts left.

Now the capital equation:

$$\dot{k} = f(k) - c - nk = Ak^\alpha - c - nk$$

Solve for $\dot{k} = 0$ locus

$$c = Ak^\alpha - nk$$

So a decrease in $A$ will result in a lower $c$ for a given $k$, but the $\dot{k} = 0$ locus still intersects the origin.
In the steady state, \( f'(k) = n + \theta \). Since changes in \( A \) do not affect \( n \) or \( \theta \), the original \( MPK_{ss} \) must be the same as the new \( MPK_{ss} \). There will be a different transition path depending on the placement of the new stable arm. Now all we have to do is fill in the middle.

Here's the idea behind any problem in which the change in policy is announced in advance: by the time that the new policy arrives, the action of the system between the announcement and the enactment need to put the system on the new stable arm. The state variable \( k \) cannot jump (unless they are seized or destroyed), and the control variable \( c \) can only jump with new information.

The question tells us that there are three possible relations between the position of the stable arm associated with the new steady state and the initial steady state.

**Case 1: The old steady state is on the new stable arm** At \( t \) when the announcement is made, there is no jump in \( c \) since we are already on the new stable arm. The system stays at the old steady state until time \( s \), at which the new dynamics take over pulling the economy along the new stable arm to the new steady state.

Since there is no change in \( k \) from \( t \) to \( s \), the MPK is constant. At time \( s \), the MPK jumps down because \( A \) jumps down (\( MPK = A \alpha k^{\alpha-1} \)). As \( k \) falls along the stable arm, MPK increases until it is back to where it was before.

**Case 2: The old steady state is above the new stable arm** Since the economy is not on the new stable arm, there must be a jump in consumption. Remember that the economy has to obey the current dynamics, so as the social planner you want to get the economy onto the new stable arm using the current dynamics. Therefore, at \( t \), \( c \) jumps down and moves to the new stable arm at time \( s \). The dynamics change at that point, and the economy rides the stable arm to the new steady state.

As \( k \) increases as the economy moves to the new stable arm, the MPK will fall. At \( s \), \( A \) jumps down, which implies that the MPK jumps down also, and then rises back to the old MPK as \( k \) decreases along the new stable arm.

**Case 3: The old steady state is below the new stable arm** Since the economy is not on the new stable arm, there must be a jump in consumption. At \( t \), \( c \) jumps up, and moves to the new stable arm at time \( s \). The dynamics change at that point, and the economy rides the stable arm to the new steady state.

As \( k \) decreases as the economy moves to the new stable arm, the MPK will rise. At \( s \), \( A \) jumps down, which implies that the MPK jumps down also, and then rises back to the old MPK as \( k \) decreases along the new stable arm.

25 Problem 22

25.1 Part A

First notice that the marginal products of the two types of capital must be equal. Otherwise, I could have invested less in one and more in the other and gotten more production. This implies that there will be the same amount of both types of capital, since each has the same exponent in the production function. This should be obvious, but for those who want to see it:

\[
\text{MPK}_1 = \text{MPK}_2 \Rightarrow \alpha k_1^{\alpha-1} k_2^\alpha = \alpha k_1^{\alpha} k_2^{\alpha-1} \Rightarrow \frac{1}{k_1} = \frac{1}{k_2} \Rightarrow k_1 = k_2
\]
This implies that we can rewrite the production function in terms of general capital $k = k_1 = k_2$.

\[ y = k^\alpha k^\alpha = k^{2\alpha} \]

From the FOC for consumption we have that in the steady state

\[ 0 = \frac{\dot{c}}{c} = f'(k) - n - \theta \Rightarrow 2\alpha k^{2\alpha - 1} = n + \theta \]

\[ k_{ss} = \left( \frac{2\alpha}{n + \theta} \right)^{\frac{1}{1-2\alpha}} \]

From the $\dot{k} = 0$ equation, we have

\[ 0 = \dot{k} = f(k) - c - nk = k^{2\alpha} - c - nk \]

\[ c_{ss} = k_{ss}^{2\alpha} - nk_{ss} = \left( \frac{2\alpha}{n + \theta} \right)^{\frac{2\alpha}{1-2\alpha}} - n \left( \frac{2\alpha}{n + \theta} \right)^{\frac{1}{1-2\alpha}} \]

### 25.2 Part B

The steady state will be on a 45-degree line from the origin, since we have that $k_1 = k_2$. When the asteroid hits, I suddenly have $k_1 > k_2 \Rightarrow \text{MPK}_1 < \text{MPK}_2$. What I would like to do in this case is take some type one capital and turn it into type two capital, so there are equal amounts of both types. However, I cannot do this, because once capital is created, it cannot be converted into another type of capital. Instead, I will invest only in type 2 capital, until $k_1$ and $k_2$ are equal. Even though there is no depreciation, when I only invest in type 2 capital, the type 1 capital will be diluted by population growth. Therefore, $k_1$ will begin to fall as $k_2$ increases. Eventually they will be equal, and then they will grow along the 45-degree line eventually back to the steady state.

### 25.3 Part C

The growth rate of consumption is

\[ \frac{\dot{c}}{c} = f'(k_1, k_2) - n - \theta = \frac{\partial f}{\partial k_2} - n - \theta = \alpha k_1^\alpha k_2^{\alpha - 1} - n - \theta \]

We have taken $r$ to be the marginal product of $k_2$, since right after the asteroid this is what we invest in.

\[ \frac{\dot{c}}{c} = \alpha k_1^\alpha k_2^{\alpha - 1} - n - \theta = \alpha \left( \frac{2\alpha}{n + \theta} \right)^{\frac{\alpha}{1-2\alpha}} \frac{1}{2} \left( \frac{2\alpha}{n + \theta} \right) \frac{\alpha - 1}{\alpha - 2} \]

\[ \frac{\dot{c}}{c} = \alpha 2^{1-a} \left( \frac{2\alpha}{n + \theta} \right)^{\frac{\alpha - 1}{1-2\alpha}} = \alpha 2^{-a} \frac{n + \theta}{2\alpha} = 2^{-\alpha} (n + \theta) \]

### 26 Problem 23

If we have a true open economy, then when the borders open to international flows of capital, the interest rate will immediately jump to $r^*$. Notice that this implies that since the interest rate determines the level of capital in the country, if $r^a \neq r^*$, then capital will instantaneously flow into or out of the country. Therefore, $k$ jumps when the economy is opened.
**Case 1:** $r^* > r^a$ In this case, $k$ jumps down, since it flows out to other countries where there is a higher marginal product of capital. This implies that net foreign assets are positive. In autarky, consumption was in a steady state.

\[
\frac{\dot{c}}{c} = \frac{1}{\sigma} (r - \theta) = 0
\]

If $r$ rises when we open the country, consumption growth will be positive.

**Case 2:** $r^* < r^a$ In this case, $k$ jumps up, since it flows in from other countries since the marginal product of capital is higher domestically. This implies that net foreign assets are negative. When $r$ falls, consumption growth will become negative.

**Case 3:** $r^* = r^a$ In this case, $k$ remains the same, net foreign assets are zero, and consumption growth is zero.

### 27 Problem 23.5

The maximization problem of a representative individual is:

\[
\max_{c_1, c_2} \ln(c_t) + \ln(c_{t+1}) \text{ subject to } c_t + \frac{c_{t+1}}{1 + r_{t+1}} = w_t + \frac{w_{t+1}}{1 + r_{t+1}}
\]

The optimal path of consumption is:

\[
c_t = \frac{1}{2} (w_t + \frac{w_{t+1}}{1 + r_{t+1}})
\]

\[
c_{t+1} = \frac{(1 + r_{t,t+1})}{2} (w_t + \frac{w_{t+1}}{1 + r_{t+1}})
\]

\[
s_t = w_t - c_t = \frac{1}{2} (w_t - \frac{w_{t+1}}{1 + r_{t+1}})
\]

From the maximization problem of the firm:

\[
w_t = (1 - \alpha)k_t^\alpha = \frac{1}{2} k_t^{0.5}
\]

\[
w_{t+1} = (1 - \alpha)k_{t+1}^\alpha = \frac{1}{2} k_{t+1}^{0.5}
\]

\[r_{t+1} = \alpha k_{t+1}^{\alpha-1} - \delta = \frac{1}{2} k_{t+1}^{-0.5} - 1
\]

We can use these equations to express the savings of the young people as a function of the capital stock in period $t$ and $t + 1$.

\[
s_t = \frac{1}{2} (w_t - \frac{w_{t+1}}{1 + r_{t,t+1}}) = \frac{1}{2} \left( \frac{1}{2} k_t^{0.5} - \frac{1}{2} k_{t+1}^{0.5} \right) = \frac{1}{2} \left( \frac{1}{2} k_t^{0.5} - k_{t+1} \right)
\]

The total capital in the economy in period $t + 1$ is equal to the total savings of the young in period $t$. Since both young and old people work, the capital per worker will be equal to:

\[
k_{t+1} = \frac{s_t}{2} = \frac{1}{8} k_t^{0.5} - \frac{1}{4} k_{t+1}
\]
At the steady state $k_t = k_{t+1} = k$.

\[ k = \frac{1}{8} k^{0.5} - \frac{1}{4} k \]

\[ k = \frac{1}{100} \]

\[ y = \frac{1}{10} \]

28 Problem 23.75

The key to this problem is noticing that according to the Leontif utility function, individuals will set $c_1 = c_2$. The budget constraint will therefore be

\[ c_1 + \frac{c_1}{1 + r_{t+1}} = w_t \Rightarrow c_1 = \frac{1 + r_{t+1}}{2 + r_{t+1}} w_t \]

This implies that first period savings are

\[ s_{1,t} = w_1 - c_{1,t} = \frac{w_t}{2 + r_{t+1}} \]

Since there is no population growth or technological change, we have

\[ k_{t+1} = s_{1,t} \]

Notice that the second period interest rate is a function of future capital

\[ r_{t+1} = \alpha k_{t+1}^{\alpha - 1} - \delta, \]

while the wage is a function of current capital

\[ w_t = (1 - \alpha) k_t^\alpha. \]

Replacing this in the equilibrium condition we get

\[ k_{t+1} = \frac{(1 - \alpha) k_t^\alpha}{2 + \alpha k_{t+1}^{\alpha - 1} - \delta} \]

In the steady state, we have $k_t = k_{t+1} = k$, so

\[ k = \frac{(1 - \alpha) k^\alpha}{2 + \alpha k^{\alpha - 1} - \delta} \]

\[ k^{1-\alpha} = \frac{(1 - \alpha)}{2 + \alpha k^{\alpha - 1} - \delta} \]

\[ (2 + \alpha k^{1-\alpha} - \delta) k^{\alpha - 1} = 1 - \alpha \]

\[ \alpha + (2 - \delta) k^{1-\alpha} = 1 - \alpha \]

\[ k = \left( \frac{1 - 2\alpha}{2 - \delta} \right)^{\frac{1}{1-\alpha}} \]
29 Problem 24

We have an OLG model with almost no parameters. A representative individual maximizes the utility from consumption when young and old taking the interest rate $r$ and the wage $w$ as given:

$$\max_{c_1, c_2} \ln c_1 + \ln c_2$$

$$c_1 + \frac{c_2}{1 + r} = w_1$$

which looks just like the standard consumption problem, in which the consumer receives income in the first period only.

The FOCS are:

$$\frac{U'(c_1)}{U'(c_2)} = \frac{c_2}{c_1} = 1 + r$$

From the budget constraint:

$$c_{1,t} + \frac{c_{1,t}(1 + r)}{(1 + r)} = w_{1,t}$$

$$c_{1,t} = s_{1,t} = \frac{w_{1,t}}{2} = a_{t+1}$$

We also know that people earn their MPL in the OLG model:

$$w_t = f(k_t) - kf'(k_t) = k^\alpha - k\alpha k^{\alpha - 1}$$

$$w_t = (1 - \alpha)k_t^\alpha$$

Substituting in the wage rate:

$$a_{t+1} = \frac{1}{2}(1 - \alpha)k_t^\alpha$$

This is an equation relating the assets held by people in the country and the capital being used in the economy.

Now we need to solve for the level of capital in the open economy. We have that

$$r^* = \text{MPK} = \alpha k^{\alpha - 1}$$

$$k_{ss} = \left(\frac{\alpha}{r^*}\right)^{\frac{1}{1 - \alpha}}$$

Plug this back into $a_{t+1}$ to get

$$a_{t+1} = \frac{1}{2}(1 - \alpha)\left(\frac{\alpha}{r^*}\right)^{\frac{\alpha}{1 - \alpha}}$$

The country will have positive exports if

$$a_{t+1} > k_{ss}$$

$$\frac{1}{2}(1 - \alpha)\left(\frac{\alpha}{r^*}\right)^{\frac{\alpha}{1 - \alpha}} > \left(\frac{\alpha}{r^*}\right)^{\frac{1}{1 - \alpha}}$$

$$\frac{1}{r^*} > \frac{1}{\alpha} \left(\frac{2}{1 - \alpha}\right)^{\frac{1}{\alpha}}$$

$$\alpha \left(\frac{1 - \alpha}{2}\right)^{\frac{1}{\alpha}} > r^*$$
30 Problem 24.5

Generally, in the OLG model, we get $k_{t+1}$ as some function of $k_t$, say $\psi(k_t)$. For example, if $\theta = 0$ and $\sigma = 1$, the individual saves half his wage (or half the marginal product). In this case, we have

$$k_{t+1} = \frac{1 - \alpha}{2} k^\alpha$$

Here $\psi(k_t)$ is concave, satisfies the inada conditions, and has $\psi(0) = 0$, so we know that it is a normal function which crosses the 45-degree line once at $k_t = 0$ and once where $k_t > 0$.

From this base case, let's see what happens when we change $\sigma > 1$. Intuitively, we should see that if we are more risk averse, we want smoother consumption between the two periods. If consumption is smooth over time, then this implies that savings will also be relatively smooth over time, and so $k_t$ will be relatively close to $k_{t+1}$. Graphically, the if $k_t$ is close to $k_{t+1}$, then this implies that $\psi(k_t)$ is close to the 45-degree line, or that $\psi(k_t)$ is more linear. Therefore, if the two countries have the same steady state, and $\psi(k_t)$ is flatter for country 2, then it must be the case that $\psi(k_t)$ for country one lies above $\psi(k_t)$ for country two before the steady state, and reverse after the steady state. Country 1 will converge faster it is steady state because its $\psi(k_t)$ is more curved.

31 Problem 25

31.1 Part A

Assume that the initial level of capital is large enough, so that the problem is well defined (meaning, $c_1 > x$).

$$\max \ln(c_1 - x) + \ln(c_2) \text{ subject to } c_1 + \frac{c_2}{1 + r} = w_1 \quad (18)$$

The first-order condition will be:

$$\frac{c_2}{c_1 - x} = 1 + r$$

Plugging this into the budget constraint we have

$$2c_1 - x = w_1 \Rightarrow$$

$$s_t = \frac{w_t - x}{2} = k_{t+1}$$

Factors are paid their marginal products, so

$$w = MPL = f(k_t) - kf'(k_t) = k^\alpha - k\alpha k^{\alpha-1} = (1 - \alpha)k_t^\alpha$$

We combine the wage and capital equations:

$$k_{t+1} = \frac{(1 - \alpha)k_t^\alpha - x}{2}$$

And we now have a differential equation (call it $\phi(k_t)$) for the evolution of capital which will determine the paths in this model, with three different cases:

1. If $x < 0$, then there is only one steady state which is stable.
2. If \( x = 0 \), there are two steady states, one at \( k = 0 \) which is unstable and one at 
\[ k = \left( \frac{1-\alpha}{2} \right)^{\frac{1}{1-\alpha}} \] which is stable.

3. If \( x > 0 \), then there can be two steady states which are both greater than zero, where one is stable and one is unstable, or one steady state with \( k_{ss} > 0 \) which is unstable, or no steady states, depending on the values of \( x \).

### 31.2 Part B

Now the \( x_t \) is endogenously determined. The FOC is the same as before because people do not take this externality into account.

Let’s start with the difference equation from above:

\[
k_{t+1} = \frac{(1 - \alpha)k_t^\alpha - c_{2,t}}{2}
\]

The old people at \( t = 2 \) are consuming the capital their savings plus the return that they earned on this savings. Therefore,

\[
c_{2,t} = s_{t-1}(1 + r) = k_t(1 + f'(k_t)) = k_t(1 + \alpha k_t^{\alpha-1})
\]

Remember that \( s_{t-1} = k_t \) because there is no population growth.

Plugging this into the difference equation

\[
k_{t+1} = \frac{(1 - \alpha)k_t^\alpha - k_t(1 + \alpha k_t^{\alpha-1})}{2} \equiv \phi(k_t)
\]

In a steady state \( k_{t+1} = k_t \implies \)

\[
k = \frac{1}{2}((1 - \alpha)k^\alpha - k(1 + \alpha k^{\alpha-1}))
\]

\[
1 = \frac{1}{2}((1 - \alpha)k^{\alpha-1} - 1 - \alpha k^{\alpha-1})
\]

\[
3 = (1 - 2\alpha)k^{\alpha-1}
\]

\[
k_{ss} = \left( \frac{3}{1 - 2\alpha} \right)^{\frac{1}{1-\alpha}} = \left( \frac{1 - 2\alpha}{3} \right)^{\frac{1}{1-\alpha}}
\]

Since \( k_{ss} < 0 \) is not valid (you can’t be continually borrowing), \( k_{ss} \) exists if

\[
\left[ \frac{1 - 2\alpha}{3} \right]^{\frac{1}{1-\alpha}} \geq 0
\]

\[
1 - 2\alpha \geq 0
\]

\[
1 - 2\alpha \geq 0
\]

\[
2\alpha \leq 1
\]

\[
\alpha \leq \frac{1}{2}
\]
\( \alpha \geq 0 \) by assumption of the production function. Therefore, \( 0 \leq \alpha \leq \frac{1}{2} \) in order for a SS to exist.

It will be stable iff \( \frac{\partial \phi(k_t)}{\partial k_t} |_{k_{ss}} < 1 \). This is a very intuitive result that follows directly from the following picture:

\[
\phi(k_t) = \frac{(1 - \alpha)k_t^\alpha - k_t(1 + \alpha k_t^{\alpha-1})}{2}
\]
\[
= \frac{1}{2}[(1 - 2\alpha)k_t^\alpha - k_t]
\]
\[
\frac{\partial \phi(k_t)}{\partial k_t} = \frac{1}{2}[(1 - 2\alpha)\alpha k_t^{\alpha-1} - 1] < 1
\]
\[
(1 - 2\alpha)\alpha k_t^{\alpha-1} - 1 < 2
\]
\[
(1 - 2\alpha)\alpha k_t^{\alpha-1} < 3
\]

Evaluate at \( k_{ss} \):

\[
(1 - 2\alpha)\alpha \left(\frac{1 - 2\alpha}{3}\right)^{\alpha-1} < 3
\]
\[
\frac{3(1 - 2\alpha)\alpha}{1 - 2\alpha} < 3
\]
\[
\alpha < 1
\]

This is a less stringent constraint that the one for the existence of a positive SS \( \Rightarrow \) as long as \( 0 \leq \alpha \leq \frac{1}{2} \) then there will be a stable SS.

### 32 Problem 26

Since the question is a per worker question, let’s put the production function in per worker terms:

\[
Y = K^{5.5}L^{5} \Rightarrow y = k^{5}
\]

Consider the consumption decision of the citizens of country 1. The maximize utility taking the interest rate \( r \) and the wage \( w \) as given:

\[
\max (1 - \gamma) \ln c_y + \gamma \ln c_o \text{ subject to } c_y + \frac{c_o}{1 + r} = w_1
\]

FOC:

\[
\frac{U'(c_y)}{U'(c_o)} = \frac{(1 - \gamma)c_o}{\gamma c_y} = 1 + r \Rightarrow
\]
\[
c_o = \frac{\gamma}{1 - \gamma} (1 + r) c_y
\]

Plugging this into the budget constraint we have:

\[
c_y + \frac{\gamma(1 + r)c_y}{1 + r} = c_y \left(1 + \frac{\gamma}{1 - \gamma}\right) = w_1 \Rightarrow
\]
\[
\frac{c_y}{1 - \gamma} = w_1 \Rightarrow c_y = (1 - \gamma) w_1 \Rightarrow s = \gamma w_1
\]
The wage is the MPL:

\[ w_t = f(k_t) - k f'(k_t) = k_t^5 - k_t(1/2)k_t^{-0.5} = \frac{1}{2}k_t^5 \Rightarrow \]

\[ s_t = w_t \gamma = \gamma \frac{1}{2} k_t^5 \]

Similarly, it will be true that the savings in country 2 will be equal to:

\[ s_t = w_t \beta = \beta \frac{1}{2} k_t^5 \]

The question remains, however, what is the level of capital in each country, \( k_{ss} \) and correspondingly, what is the interest rate. Both of these variables will be endogenously determined in the model. First, since the production functions in the 2 countries are the same, and since we have an open economy, it must be that equal amounts of capital are invested in both countries (the return on investment should be the same, i.e. the marginal product of capital should be the same; therefore, capital should be the same). The total amount of capital in the world at date \( t + 1 \) is the sum of the savings of the people in the 2 countries:

\[ 2k_{t+1} = s^1_t + s^2_t \]

\[ 2k_{t+1} = (\beta + \gamma) \frac{1}{2} k_t^5 \]

\[ k_{t+1} = (\beta + \gamma) \frac{1}{4} k_t^5 \]

This is the equation that represents the evolution of capital in each country. We can solve for the steady state level of capital:

\[ k_{ss} = (\beta + \gamma) \frac{1}{4} k_{ss}^5 \Rightarrow \]

\[ k_{ss} = \left( \frac{\beta + \gamma}{4} \right)^2 \Rightarrow \]

\[ GDP = y_{ss} = k_{ss}^{0.5} = \frac{\beta + \gamma}{4} \]

Now, we have to solve for the GNP per person in each country. Remember that the GNP is the sum of the wages of the residents plus their interest income. Since we know \( k_{ss} \), we can say right away that residents in both countries receive equal wages:

\[ w_1^{ss} = w_2^{ss} = \frac{1}{2} k_{ss}^{0.5} = \frac{\beta + \gamma}{8} \]

To determine the interest income, \( rA \), we need to know the interest rate and the level of assets in each country. The interest rate is the marginal product of capital:

\[ r = \frac{\partial y}{\partial k} = \frac{1}{2} k^{0.5} = \frac{2}{\beta + \gamma} \]

The assets of each person is the savings he does in the first period of his life. Since the residents of the two countries discount future consumption at different rates, they also have different levels of savings/assets:
\[
A^1 = s_{ss}^1 = \gamma w_{ss} = \frac{\beta + \gamma}{8}
\]
\[
A^2 = s_{ss}^2 = \beta w_{ss} = \frac{\beta + \gamma}{8}
\]

You can check that the sum of the savings in the 2 countries is equal to the total capital in the world (this should not be surprising if we did not make a mistake). Finally,

\[
\text{GNP}_1 = w + rA^1 = \frac{\beta + \gamma}{8} + \frac{2}{\beta + \gamma} \frac{\beta + \gamma}{8} = \frac{\beta + \gamma}{8} + \frac{\gamma}{4}
\]
\[
\text{GNP}_2 = w + rA^2 = \frac{\beta + \gamma}{8} + \frac{2}{\beta + \gamma} \frac{\beta + \gamma}{8} = \frac{\beta + \gamma}{8} + \frac{\beta}{4}
\]

As a final check, we can compare the sum of the GNPs and the GDPs - those should be equal as well.

33 Problem 27

33.1 Part A

Since the agents have no consumption in the first period of life, they save all that they earn in the first period of life, which is just their wage.

\[
s_t = w_t = (1 - \alpha) k^\alpha \Rightarrow
\]

Since now we have population growth, it is important to note that capital per person in the next period will savings per person divided by \(1 + n\):

\[
k_{t+1} = \frac{s_t}{1 + n} = \frac{1 - \alpha}{1 + n} k^\alpha \Rightarrow
\]
\[
k_{ss} = \left( \frac{1 - \alpha}{1 + n} \right)^\frac{1}{1 - \alpha}
\]

33.2 Part B

The golden rule tells us that the marginal product of capital equals \(n\)

\[
n = \alpha k_{ss}^{\alpha - 1}
\]

\[
\alpha \left( \frac{1 - \alpha}{1 + n} \right)^\frac{\alpha - 1}{1 - \alpha} = \alpha \frac{1 + n}{1 - \alpha} = n
\]
\[
\alpha + \alpha n = n - \alpha n
\]
\[
\alpha = (1 - 2\alpha) n \Rightarrow n = \frac{\alpha}{1 - 2\alpha} = \frac{1}{1 - \frac{2}{3}} = 1
\]
34 Problem 27.5

34.1 Part A

The agent maximizes

\[
\max_{c_1, c_2, n} \ln c_1 + \ln c_2 + \ln n \quad \text{subject to} \quad w = c_1 + n + \frac{c_2}{1 + r}
\]  

(19)

The first-order conditions of this problem are:

\[
\frac{1}{c_1} = \lambda, \quad \frac{1}{n} = \lambda, \quad \frac{1}{c_2} = \frac{\lambda}{1 + r} \Rightarrow \\
\frac{c_1}{1 + r} = n = 1
\]

Plugging this into the budget constraint we have

\[
w = c_1 + c_1 + \frac{1 + r}{1 + r} c_1 \Rightarrow \frac{w}{3} = c_1 = n \Rightarrow \\
s = \frac{w}{3}
\]

As usual the wage and the interest rate are:

\[
w_t = (1 - \alpha) k_t^\alpha \\
r_{t+1} = \alpha k_{t+1}^{\alpha-1}
\]

Suppose that there are \(L_t\) young people living at time \(t\). Their combined savings are equal to \(s_t L_t\). Each young person has \(n_t\) number of children, and consequently there are \(n_t L_t\) young people born in period \(t + 1\). The capital per person is:

\[
k_{t+1} = \frac{s_t}{n_t} = \frac{w}{3} = 1 \Rightarrow \\
k_{ss} = y_{ss} = 1 \Rightarrow \\
w = (1 - \alpha) \cdot 1^\alpha = 1 - \alpha \Rightarrow n_{ss} = \frac{1 - \alpha}{3}
\]

34.2 Part B

Notice that if \(\alpha = \frac{1}{2}\), then \(n = \frac{1}{6}\). This result is obtained through maximization. However, now consider the country in which \(n = \frac{1}{6}\) whether or not it is maximizing. The utility maximization problem looks like this:

\[
\max_{c_1, c_2} \ln c_1 + \ln c_2 + \ln \frac{1}{6} \quad \text{subject to} \quad w = c_1 + \frac{1}{6} + \frac{c_2}{1 + r}
\]  

(20)

Form the Lagrangian and find the first-order conditions:

\[
c_1 = \frac{1}{2}(w - \frac{1}{6}) \Rightarrow s = \frac{1}{2}(w - \frac{1}{6})
\]
The capital per person difference equation is:

\[ k_{t+1} = \frac{s_t}{6} = 3(w - \frac{1}{6}) = 3(\frac{1}{2}k_t^{0.5} - \frac{1}{6}) = \frac{3}{2}k_t^{0.5} - \frac{1}{2} \Rightarrow \]

\[ k = \frac{3}{2}k^{0.5} - \frac{1}{2} \Rightarrow \left(k + \frac{1}{2}\right)^2 = \frac{9k}{4} \Rightarrow k^2 - \frac{5}{4}k + \frac{1}{4} = 0 \]

\[ k = \frac{5}{4} \pm \sqrt{\frac{25}{16} - 4 \cdot \frac{1}{4}} = \frac{5}{8} \pm \frac{1}{2} \sqrt{\frac{9}{16}} = \frac{5}{8} \pm \frac{3}{8} \]

As you can see, there are two steady states: 0.25, and 1. What can you say about the speed of convergence to the steady state? While country 1 jumps directly to the steady state, capital per person in country 2 would converge asymptotically to its steady state.

35 Problem 28

35.1 Part A

\[ c_t + \frac{c_{t+1}}{1 + r} = w_t + \frac{b_{t+1}}{1 + r} \]

35.2 Part B

From the first order conditions we have

\[ c_{t+1} = (1 + r)\rho c_t \]

\[ c_t = \frac{w_t}{1 + \rho} + \frac{b_{t+1}}{(1 + \rho)(1 + r)} \]

\[ s_t = w_t - \frac{w_t}{1 + \rho} - \frac{b_{t+1}}{(1 + \rho)(1 + r)} \]

\[ s_t = \frac{\rho}{1 + \rho} w_t - \frac{b_{t+1}}{(1 + \rho)(1 + r)} \]

35.3 Part C

\[ b_{t+1} = (1 - \rho)(1 + r)k_t \]

35.4 Part D

\[ s_t = k_{t+1} = \frac{\rho}{1 + \rho} w_t - \frac{(1 - \rho)(1 + r)k_t}{(1 + \rho)(1 + r)} \]

\[ k_{t+1} = \frac{\rho}{1 + \rho} (1 - \alpha)k_t^\alpha - \frac{(1 - \rho)(1 + r)k_t}{(1 + \rho)(1 + r)} \]

\[ k_{t+1} = \frac{\rho}{1 + \rho} (1 - \alpha)k_t^\alpha - \frac{(1 - \rho)k_t}{(1 + \rho)} \]
Steady state is when
\[
k = \frac{\rho}{1 + \rho} (1 - \alpha)k^\alpha - \frac{(1 - \rho)}{(1 + \rho)} k
\]
\[
\frac{2}{1 + \rho} k = \frac{\rho}{1 + \rho} (1 - \alpha)k^\alpha
\]
\[2k^{1-\alpha} = \rho(1 - \alpha)\]
\[
k_{ss} = \left(\frac{\rho(1 - \alpha)}{2}\right)^{\frac{1}{1-\alpha}}
\]
Notice that this is the steady state level of capital we have without the probability of death, except that the inside is multiplied by $\rho$.

35.5 Part E

Taking natural logs we have
\[
\ln (k_{ss}) = \frac{1}{1 - \alpha} [\ln (\rho) + \ln (1 - \alpha) - \ln (2)] \Rightarrow
\]
\[
\frac{\partial k_{ss}}{\partial \rho} = \frac{1}{(1 - \alpha)\rho} > 0
\]
Therefore, if the probability of surviving decreases, the steady state capital stock will decrease.

36 Problem 29

36.1 Part A

\[
\max_{c_1, c_2} U = \ln c_1 + \ln c_2 \text{ subject to } c_1 + \frac{c_2}{1 + r} = w_1 \tag{21}
\]

From the first order conditions and the budget constraint we have
\[
c_1 = \frac{w}{2} \Rightarrow s = \frac{w}{2} \Rightarrow
\]
\[
k_{t+1} = \frac{s_t}{1 + n} = \frac{w_t}{2(1 + n)}
\]
Factors are paid their marginal product:
\[
w_t = f(k) - kf'(k) = k_t^\alpha - \alpha k_t^\alpha = (1 - \alpha)k_t^\alpha
\]
Therefore,
\[
k_{t+1} = \frac{(1 - \alpha)k_t^\alpha}{2(1 + n)} \Rightarrow
\]
\[
k = \frac{(1 - \alpha)k^\alpha}{2(1 + n)} \Rightarrow k^{1-\alpha} = \frac{(1 - \alpha)}{2(1 + n)} \Rightarrow
\]
\[
k_{ss} = \left(\frac{1 - \alpha}{2(1 + n)}\right)^{\frac{1}{1-\alpha}}
\]
36.2 Part B

We now solve for $c_1$ in the steady state:

$$c_1 = s_1 = \frac{w_1}{2} = \frac{(1-\alpha)k^\alpha}{2} = \frac{1-\alpha}{2} \left( \frac{1-\alpha}{2(1+n)} \right)^{\frac{\alpha}{1-\alpha}} = \left( \frac{1-\alpha}{2} \right)^{\frac{1}{1-\alpha}} \left( \frac{1}{1+n} \right)^{\frac{\alpha}{1-\alpha}}$$

Remember that in a closed economy

$$r = MPK - \delta = \alpha k^{\alpha-1} - 1$$

And now we solve for $c_2$

$$c_2 = s(1+r) = \left( \frac{1-\alpha}{2} \right)^{\frac{1}{1-\alpha}} \left( \frac{1}{1+n} \right)^{\frac{\alpha}{1-\alpha}} a k^{\alpha-1}$$

$$c_2 = \left( \frac{1-\alpha}{2} \right)^{\frac{1}{1-\alpha}} \left( \frac{1}{1+n} \right)^{\frac{\alpha}{1-\alpha}} \left[ \alpha \left( \frac{1-\alpha}{2(1+n)} \right)^{\frac{\alpha-1}{1-\alpha}} \right]$$

$$= \alpha \left( \frac{1-\alpha}{2} \right)^{\frac{\alpha}{1-\alpha}} (1+n)^{\frac{1-2\alpha}{1-\alpha}}$$

36.3 Part C

$$U_{ss} = \ln c_{1,ss} + \ln c_{2,ss} =$$

$$= \ln \left( \left( \frac{1-\alpha}{2} \right)^{\frac{1}{1-\alpha}} \left( \frac{1}{1+n} \right)^{\frac{\alpha}{1-\alpha}} \right) + \ln \left( \alpha \left( \frac{1-\alpha}{2} \right)^{\frac{\alpha}{1-\alpha}} (1+n)^{\frac{1-2\alpha}{1-\alpha}} \right) =$$

$$= \frac{1}{1-\alpha} \ln \left( \frac{1-\alpha}{2} \right) - \frac{\alpha}{1-\alpha} \ln (1+n) + \frac{\alpha}{1-\alpha} \ln \left( \frac{1-\alpha}{2} \right) + \ln(\alpha) + \frac{1-2\alpha}{1-\alpha} \ln (1+n)$$

$$= \frac{\alpha + 1}{1-\alpha} \ln \left( \frac{1-\alpha}{2} \right) + \frac{1-3\alpha}{1-\alpha} \ln (1+n) + \ln(\alpha)$$

$$= \ln \left[ \alpha \left( \frac{1-\alpha}{2} \right)^{\frac{\alpha+1}{1-\alpha}} (1+n)^{\frac{1-3\alpha}{1-\alpha}} \right]$$

36.4 Part D

$$\alpha = \frac{1}{2}$$

$$U_{ss} = \ln \left[ \alpha \left( \frac{1-\alpha}{2} \right)^{\frac{\alpha+1}{1-\alpha}} (1+n)^{\frac{1-3\alpha}{1-\alpha}} \right] = \ln \left[ \frac{1}{2} \left( \frac{1}{2} \right)^{\frac{3}{2}} (1+n)^{\frac{-1}{2}} \right]$$

$$= -\ln (128) - \ln (1+n)$$

$$\frac{\partial U_{ss}}{\partial n} = \frac{-1}{1+n}$$

Since $n$ is bounded below by $-1$, the denominator is weakly positive, and therefore the whole expression can never be positive. Therefore, if $n$ increases, $U_{ss}$ falls.
Problem 29.5

From the consumer’s optimization problem we know that savings are exactly one half of first-period earnings. First of all, we find the change in capital stock when technology is not changing between periods. The evolution of the capital stock in the economy will be described by:

\[ k_{t+1} = \frac{1}{2} w_t = \frac{1}{2} (1 - \alpha) k_t^\alpha \]

When the technology level does fluctuate between periods, we have that

\[ k_{t+1} = \frac{w_t}{2(1 + \gamma)} = \frac{1 - \alpha}{2(1 + \gamma)} k_t^\alpha \]

In the steady state, the capital stock will fluctuate between two levels, such that

\[ k_{ss, odd} = \frac{1}{2} (1 - \alpha)(k_{even, ss})^\alpha \]

\[ k_{ss, even} = \frac{(1 - \alpha)(k_{odd, ss})^\alpha}{2(1 + \gamma)} \]

Therefore,

\[ k_{ss, odd} = \left[ \frac{1}{1 + \gamma} (\frac{1 - \alpha}{2})^{1+\alpha} \right] \frac{1}{1 - \alpha^2} \]

\[ k_{ss, even} = \frac{1 - \alpha}{2} \left[ \frac{1}{1 + \gamma} (\frac{1 - \alpha}{2})^{1+\alpha} \right] \frac{\alpha}{1 - \alpha^2} \]

Transitional Dynamics: draw a diagram picturing the first 2 equations with capital today on the horizontal axis, and capital tomorrow on the vertical. Both graphs will cross the 45 degree line (why?). Denote the 2 points of intersection \( k_A \) and \( k_B \) \( (k_A < k_B) \). Convince yourself that if \( k_0 \) is smaller than \( k_A \), capital will be growing, and if \( k_0 \) is larger than \( k_B \), capital will be falling.

38 Problem 30

This question is going to be dealing with efficiency units, and therefore we need to be very explicit with use of upper and lower case letters. Lower case letters will mean "per efficiency unit", not per capita as they have in the previous questions.

From the first order conditions, we have that:

\[ \frac{S_t}{L_t} = \frac{W_t}{2L_t} \]

Notic that \( n = 0 \) implies that \( L_t = L \) for all \( t \).

From the OLG model we have

\[ \frac{A_{t+1}}{L} = \frac{S_t}{L} = \frac{W_t}{2L} \]

and

\[ \frac{A_t}{L} = \frac{W_{t-1}}{2L} \]
which implies

\[
\frac{A_t}{e_t L} = \frac{W_{t-1}}{2e_{t-1} L} \cdot \frac{e_{t-1}}{e_t}
\]

From the given of the problem we know that:

\[
et_{t+1} = (1 + g)e_t
\]

\[
et_{t-1} = \frac{1}{1 + g}
\]

Therefore, plugging this into the asset equation

\[
\frac{A_t}{e_t L} = \frac{W_{t-1}}{2e_{t-1} L(1 + g)}
\]

\[
a_t = \frac{w_{t-1}}{2(1 + g)}
\]

From the production function we have:

\[
y_t = k_t^{\frac{1}{2}}
\]

Therefore,

\[
w_t = (1 - \frac{1}{2})k_{t-1}^{\frac{1}{2}} = \frac{1}{2}k_{t-1}^{\frac{1}{2}}
\]

Therefore, plugging this into the asset equation

\[
a_t = \frac{\frac{1}{2}k_{t-1}^{\frac{1}{2}}}{2(1 + g)} = \frac{k_{t-1}^{\frac{1}{2}}}{4(1 + g)}
\]

Assets per efficiency unit are constant in the steady state, but technology is increasing at rate \(g\), and therefore assets per capita growing are growing in the steady state.

In the open economy \(r\) is taken as given:

\[
r = MPK = \frac{1}{2}k_{ss}^{-\frac{1}{2}} \Rightarrow
\]

\[
k_{ss} = \frac{1}{4r^2}
\]

The economy is always at \(k_{ss}\) because it is an open economy. Therefore, combining \(k_{ss}\) and \(a_t\)

\[
a_{ss} = \left(\frac{1}{4r^2}\right)^\frac{1}{2} = \frac{1}{8r(1 + g)}
\]

The economy will have no foreign assets iff

\[
a_{ss} = k_{ss} \Rightarrow \frac{1}{8r(1 + g)} = \frac{1}{4r^2} \Rightarrow
\]

\[
r = 2(1 + g)
\]